# Combinatorial topology and anima

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# Contents


**1 Higher groupoids** **5**
- 1.1 Introduction . . . 5
- 1.2 Groupoid-based mathematics . . . 5
  - 1.2.1 The definition of a groupoid . . . 5
  - 1.2.2 The underlying set of a groupoid . . . 7
- 1.3 Going beyond groupoids . . . 8
  - 1.3.1 Tor functors . . . 8
  - 1.3.2 A groupoid of reasons . . . 9
- 1.4 Simplicial sets and $\infty$-groupoids . . . 11

**I Simplicial sets** **13**

**2 Simplicial sets** **14**
- 2.1 Semisimplicial sets . . . 14
- 2.2 Simplicial sets . . . 17
- 2.3 Some basic examples of simplicial sets . . . 22
  - 2.3.1 Simplices . . . 22
  - 2.3.2 The nerve of a poset . . . 22
  - 2.3.3 0-dimensional simplicial sets . . . 23
  - 2.3.4 Simplicial subsets . . . 23
- 2.4 Filling in a simplicial object . . . 24
  - 2.4.1 Dimension . . . 27
  - 2.4.2 The skeletal filtration . . . 28
  - 2.4.3 Glueing simplices . . . 30
- 2.5 Braced simplicial sets . . . 31
- 2.6 Connected components . . . 34
- 2.7 Path components . . . 36

**3 Geometric realization** **38**
- 3.1 Introduction . . . 38
- 3.2 Presheaf categories as free cocompletions . . . 38
- 3.3 Constructing geometric realization . . . 40
- 3.4 The simplicial set associated to a topological space . . . 41
- 3.5 The connected components of a geometric realization . . . 42

3.6 Geometric realizations and products . . . 44
3.6.1 Drinfeld's construction of the geometric realization . . . 45
3.6.2 Applications to finite limits . . . 47
3.7 The nerve of a category . . . 47

**II $\infty$-groupoids 50**

**4 The definition 51**
4.1 Introduction . . . 51
4.2 The essential image of $N$ . . . 51
4.2.1 Categories . . . 51
4.2.2 Groupoids . . . 53
4.3 Strict 2-groupoids as simplicial sets . . . 55
4.4 Embedding $\infty$-groupoids into simplicial sets . . . 59
4.5 Equivalence of $\infty$-groupoids . . . 60
4.5.1 Mapping spaces . . . 61
4.5.2 Homotopy via mapping spaces . . . 63
4.5.3 Digression: Homotopies of continuous functions . . . 63
4.6 The $\pi_0$ of an $\infty$-groupoid . . . 65
4.7 Absolutely anodyne maps . . . 66
4.7.1 Strongly anodyne morphisms . . . 67
4.8 Kan fibrations . . . 69
4.8.1 Anodyne morphisms . . . 69
4.8.2 A remarkable uniqueness . . . 70
4.8.3 Exponentiation for Kan fibrations . . . 71
4.8.4 Lifting in families . . . 73
4.9 Homotopy of functors between $\infty$-groupoids . . . 73
4.10 A homotopy Yoneda lemma . . . 74

**5 Homotopy fibers 76**
5.1 Introduction . . . 76
5.2 Homotopy fibers . . . 76
5.2.1 Strict fibers . . . 76
5.2.2 Digression: pointed mapping spaces . . . 77
5.2.3 Homotopy fibers . . . 78
5.3 Fibers of Kan fibrations . . . 79
5.3.1 Locally trivial fibrations . . . 83
5.4 Homotopy fibers as strict fibers . . . 85
5.5 Fibrant replacement . . . 87

**6 Morphisms and composition in an $\infty$-groupoid 90**
6.1 Introduction . . . 90
6.2 Morphism spaces . . . 90
6.3 Homotopy invariance of morphisms . . . 91
6.4 2-morphisms . . . 93
6.5 $n$-morphisms in general . . . 95

6.5.1 Left morphisms . . . 95
6.5.2 Fibers and retractions . . . 97
6.5.3 Constructing $r_n$ and proving a map anodyne . . . 98
6.5.4 Finishing our argument . . . 101
6.6 Composition of $n$-morphisms . . . 102
6.6.1 Composition of 1-morphisms . . . 102
6.6.2 The fundamental groupoid . . . 105
6.6.3 Higher morphisms . . . 108

**7 The long exact sequence of a fibration** **109**
7.1 Introduction . . . 109
7.2 The monodromy sequence in the case of a Kan fibration . . . 112
7.3 Application: The fundamental group of a circle . . . 112
7.3.1 Simplicial groups . . . 112
7.3.2 The fundamental group of the circle . . . 114
7.4 Application: Path lifting with endpoints . . . 115
7.5 Continuing the monodromy sequence . . . 116
7.6 The higher homotopy groups . . . 118
7.7 The long exact sequence of the homotopy fiber . . . 122

**8 Weak equivalences** **124**
8.1 Introduction . . . 124
8.2 Basic properties of weak equivalences . . . 125
8.3 Some examples of weak equivalences . . . 127
8.4 Weak equivalences and inclusions . . . 128
8.5 Fibrant replacement, redux . . . 129
8.6 Geometric realization of weak equivalences . . . 133

**9 Whitehead's theorem** **134**
9.1 Introduction . . . 134
9.2 Higher connectivity . . . 135
9.3 Relative $n$-connectivity . . . 137
9.4 Whitehead's theorem for functors . . . 138
9.5 Families of homotopy equivalences . . . 139

**10 Homotopy pullbacks and pushouts** **143**
10.1 Introduction . . . 143
10.2 Basic properties of the homotopy fiber product . . . 143
10.3 Homotopy pushouts . . . 146

## III Advanced properties of simplicial sets 153

**11 Barycentric subdivision** **154**
11.1 Introduction . . . 154
11.2 Barycentric subdivision . . . 155
11.2.1 The subdivision of a poset . . . 157

11.2.2 The subdivision of a braced simplicial set . . . 158
11.3 The geometric realization of a subdivision . . . 159
11.4 Simplicial approximation for posets . . . 162
11.4.1 On nerves of posets . . . 162
11.4.2 Creating homotopies . . . 164
11.4.3 Constructing our monotone map . . . 164
11.5 The last vertex map . . . 168
11.5.1 Anodyne applications . . . 170
11.6 Simplicial approximation for braced simplicial sets . . . 173

**12 Kan's $\mathrm{Ex}^\infty$-functor** **174**
12.1 Introduction . . . 174
12.2 The adjunction preserves homotopies . . . 176
12.3 $\mathrm{Ex}^\infty(T)$ is always Kan . . . 179
12.4 Using $\mathrm{Ex}^\infty$ as fibrant replacement . . . 181
12.4.1 Trivial Kan fibrations . . . 182
12.4.2 Structure of anodyne maps . . . 183
12.5 Excision . . . 184
12.5.1 A comparison result . . . 184
12.5.2 Excision . . . 187

**13 Milnor's theorem and simplicial approximation** **188**
13.1 Introduction . . . 188
13.2 The proof of Milnor's theorem . . . 188
13.2.1 Finite simplicial sets . . . 189
13.2.2 Finishing . . . 191
13.3 Some applications . . . 191
13.3.1 Geometric realization detects weak equivalence . . . 191
13.3.2 Simplicial approximation . . . 192

**A Producing anodyne morphisms** **193**
A.1 Introduction . . . 193
A.2 Digression: opposites . . . 193
A.3 Leibniz product . . . 194
A.4 Stability properties of Leibniz products . . . 196
A.5 Prisms . . . 198
A.5.1 The simplest boundary inclusion . . . 199
A.5.2 Finishing the simplest horn . . . 200
A.5.3 Harder horns . . . 201
A.6 Concluding . . . 202

# Chapter 1

# Higher groupoids

## 1.1 Introduction

In mathematics, one often encounters the quotient of a set by an equivalence relation. Sometimes, though, there can be multiple 'reasons' that two elements are equivalent.

**Example 1.1.1.** Let $\mathbb{R}^\times$ act on $\mathbb{R}$ by scaling. Define $x \sim y$ if there is some $t \in \mathbb{R}^\times$ such that $y = tx$. Then $1 \sim 2$ for exactly one reason (the reason $t = 2$), but $0 \sim 0$ for infinitely many reasons: any $t \in \mathbb{R}^\times$ gives a different proof that $0 \sim 0$.

The standard formalism of equivalence relations is not built to track 'reasons' for equivalence. In Example 1.1.1, this appears relatively harmless; however, in other examples, forgetting that elements can be equivalent for multiple reasons can cause trouble.

**Example 1.1.2.** Let $X = \{0, 1\}^3$. Consider the group $G = S_3$, acting by permuting the bits of a string in $X$.

One can define an equivalence relation $\sim$ on $X$ by saying $x \sim y$ whenever there is some $g \in G$ so that $g \cdot x = y$. Naively, we might expect the resulting quotient set $X/\sim$ to have cardinality $|X|/|G|$. But $|X|/|G| = 8/6 = 4/3$, which is not even an integer! What is going on?

The trouble is that there are some pairs $(x, y)$ for which *multiple* $g \in G$ solve $g \cdot x = y$. Thus two elements $x, y$ can be equivalent for *multiple reasons*: in our binary string example, we have $010 \sim 001$ for *two reasons*: the permutations (23) and (123) *both* send 010 to 001.

If one wants to calculate the number of $G$-orbits correctly, they need to understand this phenomenon; the correct formula for $|X/G|$ is often dubbed *Burnside's lemma*, though we will not need it.

To handle situations like this, it is helpful to replace equivalence relations by *groupoids*.

## 1.2 Groupoid-based mathematics

### 1.2.1 The definition of a groupoid

Before defining groupoids, it is helpful to recall the formal definition of an equivalence relation.

**Definition 1.2.1.** An *equivalence relation* on a set $X$ is a subset $E \subseteq X \times X$ such that

1. $E$ is reflexive: if $x \in X$, then $(x, x) \in E$;
2. $E$ is transitive: if $(x, y), (y, z) \in E$, then $(x, z) \in E$;
3. $E$ is symmetric: if $(x, y) \in E$, then $(y, x) \in E$.

**Ideation 1.2.2.** The equivalence relation in Example 1.1.2 was defined by saying "$x \sim y$ if *there exists* a $g$ with $gx = y$." The phrase *there exists* implies that this equivalence relation loses information: it doesn't remember *which* $g \in G$ witnessed the equivalence, just that there does exist one witness.

In the definition of groupoid, we replace the set $E$ of pairs of equivalent elements with a set of *reasons* for equivalence.

**Definition 1.2.3.** A *groupoid* on a set $X$ consists of

1. a set $E$,
2. two functions $s, t : E \to X$ (source and target),
3. a function $\mathrm{id} : X \to E$ (this plays the role of reflexivity: $\mathrm{id}(x)$ is some reason that $x \sim x$),
4. a binary operation $\mu : (E, t) \times_X (E, s) \to E$ (this plays the role of transitivity: if we have a reason $e$ that $x \sim y$ and a reason $f$ that $y \sim z$, then $\mu(e, f)$ is a reason that $x \sim z$),
5. a map $\iota : E \to E$ (this plays the role of symmetry: if $e$ is a reason for $x \sim y$, then $\iota(e)$ is a reason for $y \sim x$),

such that

1. $s(\mathrm{id}(x)) = t(\mathrm{id}(x)) = x$,
2. $s(\mu(e, f)) = s(e)$ and $t(\mu(e, f)) = t(f)$,
3. $s(\iota(e)) = t(e)$ and $t(\iota(e)) = s(e)$,
4. $\mu$ is associative:
$$\mu(e_1, \mu(e_2, e_3)) = \mu(\mu(e_1, e_2), e_3),$$
5. $\mu$ is unital:
$$\mu(e, \mathrm{id}_{t(e)}) = \mu(\mathrm{id}_{s(e)}, e) = e,$$
6. $\iota$ and $\mu$ are inverse:
$$\mu(e, \iota(e)) = \mathrm{id}_{s(e)},$$
$$\mu(\iota(e), e) = \mathrm{id}_{t(e)}\,.$$

**Example 1.2.4** (Quotient groupoid of a group action)**.** Let $X$ be a set and $G$ a group acting on $X$. We can define the *quotient groupoid* $[X/G]$ to be the groupoid on $X$ where

1. $E = G \times X$,
2. $s : E \to X$ is $s(g, x) = x$,

3. $t : E \to X$ is $t(g, x) = g \cdot x$,
4. $\mathrm{id} : X \to E$ is $\mathrm{id}(x) = (1, x)$,
5. the composition $\mu$ is $\mu((g, x), (h, gx)) = (hg, x)$,
6. the inverse $\iota$ is $\iota(g, x) = (g^{-1}, gx)$.

**Definition 1.2.5.** A *morphism* of groupoids

$$F : (X, E) \to (X', E')$$

is the data of

1. a function $F : X \to X'$,
2. a function $F : E \to E'$,

such that

1. $s(F(e)) = F(s(e))$,
2. $t(F(e)) = F(t(e))$,
3. $\mu(F(e), F(e')) = F(\mu(e, e'))$,
4. $F(\mathrm{id}(x)) = \mathrm{id}(F(x))$.

**Remark 1.2.6.** It's easy to deduce that $F(\iota(e)) = \iota(F(e))$.

**Remark 1.2.7.** A groupoid can equivalently be defined as a category $\mathcal{G}$ in which every arrow is an isomorphism: the set $X$ above is the set of objects of $\mathcal{G}$, with $E$ the set of morphisms; the functions $\mu, s, t$ encode composition, source, and target of morphisms; $\iota$ sends a morphism to its inverse. Morphisms of groupoids are then just functors.

We will often use standard category theory notation when dealing with groupoids: so we write $x \in \mathcal{G}$ to mean $x$ is an object, $a : x \to x'$ to mean $a$ is an arrow with source $x$ and target $x'$, etc.

### 1.2.2 The underlying set of a groupoid

One should think that a groupoid $\mathcal{G}$ is a fancy model of the quotient of a set by an equivalence relation. However, the quotient of a set by an equivalence relation *is* still a useful notion, and so we often want to pass from a groupoid to the quotient set.

**Definition 1.2.8.** The *underlying set* of a groupoid $\mathcal{G} = (X, E, s, t, \mu, \iota)$ is the quotient set $X/\sim$, where

$$(x \sim y) \iff (\exists e \in E, s(e) = x, t(e) = y).$$

We denote the underlying set of $\mathcal{G}$ by $\pi_0(\mathcal{G})$.

**Remark 1.2.9.** Equivalently, $\pi_0(\mathcal{G})$ is the set of *isomorphism classes* of objects in the category $\mathcal{G}$.

## 1.3 Going beyond groupoids

### 1.3.1 Tor functors

Let's introduce another example where remembering the 'reasons' for two elements of a quotient to be equal is helpful.

Set $A = \mathbb{C}[x, y]$, and let $M$ be an $A$-module. One can create an $A$-module $M/xM$ by quotienting $M$ by the action of $x$. In $M/xM$, we set $m = m'$ whenever

$$m' - m \in xM,$$

or, in other words, whenever we can find some $n \in M$ so that

$$m' - m = xn.$$

As a groupoid, we can present this as

$$M \times M \rightrightarrows M,$$

with $(m, n)$ mapping to $m$ and $m + xn$.

In particular, the automorphism group of $m \in M$ (viewed as an object of our groupoid) is just the set $\{n \in M \mid xn = 0\}$. In other words, the $x$-torsion of $M$.

Those familiar with homological algebra should observe that

$$\mathrm{Tor}_0^A(A/x, M) = M/xM,$$

and $\mathrm{Tor}_1^A(A/x, M)$ is the $x$-torsion of $M$. So, in this example, the groupoid quotient 'sees' the Tor functors of homological algebra (which have many uses!).

But homological algebra does not just have $\mathrm{Tor}_0$ and $\mathrm{Tor}_1$; there can be *higher* Tor functors. For example, let's suppose we had instead formed the quotient $M/(x, y)M$. To model this as a groupoid, it seems natural to use the maps

$$M \times M \times M \rightrightarrows M,$$

sending $(m, n_1, n_2)$ to $(m, m + xn_1 + yn_2)$. As before, the automorphism group of $m \in M$ is just

$$\{(n_1, n_2) \in M^2 \mid xn_1 + yn_2 = 0\}.$$

But what are the Tor functors here? To compute these, use the free resolution

$$A \xrightarrow{(-y,x)} A \oplus A \xrightarrow{(x,y)} A$$

of $A/(x, y)$.

Tensoring this with $M$, we find that

$$\mathrm{Tor}_0^A(A/(x, y), M) = M/(x, y)M,$$

$$\mathrm{Tor}_1^A(A/(x, y), M) = \{(n_1, n_2) \in M^2 \mid xn_1 + yn_2 = 0\}/\{(-yn, xn) \mid n \in M\},$$

$$\mathrm{Tor}_2^A(A/(x, y), M) = \{n \in M \mid xn = yn = 0\}.$$

This is strange: the automorphism groups of our groupoid are *too large*, and our groupoid doesn't see the $\mathrm{Tor}_2$ at all.

There is an easy way to explain this discrepancy: our groupoid quotient comes, essentially, from tensoring

$$A^2 \xrightarrow{(x,y)} A$$

with $M$, whereas the Tor functors are computed by tensoring the 3-term complex

$$A \xrightarrow{(-y,x)} A^2 \xrightarrow{(x,y)} A$$

with $M$.

These two complexes

$$A^2 \xrightarrow{(x,y)} A$$

and

$$A \xrightarrow{(-y,x)} A^2 \xrightarrow{(x,y)} A$$

are different in a subtle way.

The complex $A^2 \xrightarrow{(x,y)} A$ is just a presentation of the $A$-module $A/(x,y)$. In this presentation, $xy$ is set to zero for *two* reasons: once because it's $x \cdot y$, and $y$ was set to zero; and a second time because it's $y \cdot x$, and $x$ was set to zero.

But when we consider the free resolution

$$A \xrightarrow{(-y,x)} A^2 \xrightarrow{(x,y)} A,$$

we are *forcing those two 'reasons' that $xy = 0$ to coincide*. Both complexes are describing the quotient $A/(x,y)$, but in the two complexes we have a different number of *reasons* that $xy = 0$.

As homological algebra has proven itself to have many wonderful applications, it is natural to ask if we can encode this 3-term complex using groupoids. Unfortunately, this is a major problem for groupoid-based mathematics: while we can easily encode 2-term complexes with groupoids, these more complicated complexes with interesting syzygies are not encodable via groupoids: groupoids only encode reasons that elements are equivalent, not reasons that *reasons* are equivalent!

### 1.3.2 A groupoid of reasons

How do we make groupoids more expressive?

If one really embraces doing mathematics with groupoids, one might notice something interesting: the definition of groupoid uses *sets*, when perhaps it should use *groupoids*. If we really want to redo mathematics using groupoids in place of sets, then we should re-define groupoids using groupoids!

This leads to the idea of *2-groupoids*.

**Notation 1.3.1.** Let $f : \mathcal{G}_1 \to \mathcal{G}_0, f' : \mathcal{G}'_1 \to \mathcal{G}_0$ be two morphisms of groupoids. We define the *strict fiber product* $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}'_1$ to be the groupoid of pairs $(x \in \mathcal{G}_1, y \in \mathcal{G}'_1)$ such that $f(x) = f'(y)$, with morphisms $(x,y) \to (x',y')$ being pairs of maps $(i : x \to x', j : y \to y')$ such that $f(i) = f'(j)$.

**Definition 1.3.2.** A *groupoid object in the category of groupoids* is the data of

1. a groupoid $\mathcal{G}_0$ (of objects),
2. a groupoid $\mathcal{G}_1$ (of arrows),

3. two functors $s, t : \mathcal{G}_1 \to \mathcal{G}_0$,
4. a functor $\mu : (\mathcal{G}_1, t) \times_{\mathcal{G}_0} (\mathcal{G}_1, s) \to \mathcal{G}_1$,
5. a functor $e : \mathcal{G}_0 \to \mathcal{G}_1$,
6. a functor $\iota : \mathcal{G}_1 \to \mathcal{G}_1$,

such that

1. $s \circ \mu$ is equal to $s \circ p_1$, for $p_1 : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ the projection onto the first factor,
2. $t \circ \mu$ is equal to $t \circ p_2$,
3. $s \circ e = t \circ e = \mathrm{id}_{\mathcal{G}_0}$,
4. the functors
$$f \mapsto \mu(f, e(t(f))),$$
$$f \mapsto \mu(e(s(f)), f)$$
are both equal to $\mathrm{id}_{\mathcal{G}_1}$,
5. $s \circ \iota = t, t \circ \iota = s$,
6. the functors $e \circ s$ and
$$f \mapsto \mu(f, \iota(f))$$
are equal,
7. the functors $e \circ t$ and
$$f \mapsto \mu(\iota(f), f)$$
are equal,
8. composition is associative: the two functors
$$(\mathcal{G}_1, t) \times_{\mathcal{G}_0} (s, \mathcal{G}_1, t) \times_{\mathcal{G}_0} (s, \mathcal{G}_1) \to \mathcal{G}_1$$
$$(a, b, c) \mapsto \mu(a, \mu(b, c))$$
$$(a, b, c) \mapsto \mu(\mu(a, b), c)$$
are equal.

**Remark 1.3.3.** We can think of an ordinary groupoid as starting with a set of objects, and then attaching isomorphisms between them.

A groupoid object in the category of groupoids can similarly be viewed as starting with a groupoid $\mathcal{G}_0$, and then attaching both *new morphisms* to our groupoid (these are the objects of $\mathcal{G}_1$), and attaching *morphisms between morphisms* (these are the arrows of $\mathcal{G}_1$).

In this interpretation, it is a little annoying that $\mathcal{G}_0$ is already a groupoid, instead of merely being a set, as it means we have two types of morphisms: the ones which come from $\mathcal{G}_0$, and the ones which come from $\mathcal{G}_1$. To avoid this, we make the following definition.

**Definition 1.3.4.** A *strict* 2-*groupoid* is a groupoid object $\mathcal{G}_\bullet$ in the category of groupoids, whose object groupoid $\mathcal{G}_0$ is a *set*: that is, every morphism of $\mathcal{G}_0$ is an identity morphism.

**Remark 1.3.5.** These objects are called *strict* 2-groupoids because, in our definition of 2-groupoids, we required the functors $\mu, s, t$, etc. to obey certain *equalities*, instead of mere natural isomorphisms.

There are other possible formalizations of the idea of a 2-groupoid, which are laxer in how strictly the axioms are enforced. For the moment, we will not explore these laxer notions, but we point out their existence to emphasize the headaches of higher category theory: already for 2-groupoids, there's some trouble deciding what the correct definition should be!

If you take groupoid-based mathematics seriously, you should realize that groupoids are inadequate, and transition to 2-groupoid-based mathematics.

Unfortunately, it doesn't stop here: if you deal with 2-groupoids long enough, you'll realize that you should instead use something like "groupoid objects in the category of 2-groupoids," leading you to study *3-groupoids*; and if you take 3-groupoids seriously, then you'll realize the correct notion should be something like a groupoid object in the category of 3-groupoids, and so on.

The goal of these notes is to explain how to deal with the limiting object: $\infty$-*groupoids*.

There are many, many possible definitions of $\infty$-groupoid, depending on how strictly you enforce your axioms. One needs to strike a balance: many naturally occurring examples of $\infty$-groupoids do not have strict associativity – the compositions $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are not always literally equal, but instead just isomorphic – so having strict axioms makes it hard for your theory to accommodate examples; but the less strict your axioms are, the more headaches you get when trying to prove basic results about $\infty$-groupoids. Decades of mathematical practice have revealed *Kan complexes* as the most practical model of $\infty$-groupoids: they are strict enough that it is easy to prove theorems about them, but lax enough that all natural examples are easily encoded as $\infty$-groupoids.

## 1.4 Simplicial sets and $\infty$-groupoids

Independently of the algebraists, the geometers and algebraic topologists were exploring shapes. The standard notion of a point-set topological space is often incredibly convenient, but for studying more combinatorial properties of shapes it can be a bit cumbersome. For example, formalizing arguments about triangulations – like a proof of Euler's polyhedral formula – using point-set topological spaces is often a pain, as various point-set pathologies can distract from geometric considerations. For this reason, algebraic topologists invented a different formalization of shape, emphasizing the more combinatorial properties.

Recall that the ancient Greeks underwent one of the first systematic studies of shape, studying the Euclidean plane. When doing Euclidean geometry in the style of the ancient Greeks, you do not think of the plane as being made of an infinitude of points; instead, you name a few points on your diagram and only access those. This is in stark contrast to the point-set topological approach, where you need to deal with all the points at once.

*Simplicial sets* are a model for the idea of shape that takes the Greek approach: roughly, a simplicial set is a set of 'lego pieces' of various dimensions (some points, some edges, and some higher dimensional pieces), together with instructions on how to stick them together: this edge should go from point $P$ to point $Q$, this 2-dimensional piece has line $\ell$ as one of its sides, etc.

As a baby example, the circle can be modelled as a simplicial set in (at least!) two ways.

**Example 1.4.1** (The simplicial circle)**.** Our first 'simplicial circle' will consist of a set

$$C_0 = \{P, Q\}$$

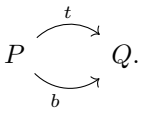
of two points $P$ and $Q$, and a set

$$C_1 = \{t, b\}$$

of edges, together with the following glueing instructions:

1. $t$ is an edge from $P$ to $Q$,
2. $b$ is also an edge from $P$ to $Q$.

If we take two edges and two points, and glue them together with those instructions, we get a circle; think of $t, b$ as being the top and bottom half.




**Example 1.4.2** (The simplicial circle, again)**.** We can build the circle in another way. Let

$$D_0 = \{P\}$$

be a single point, and $D_1 = \{\ell\}$ a set consisting of a single edge. We glue these together with one instruction: $\ell$ is an edge from $P$ to $P$.




But why should one bother to learn the simplicial set formalism after already becoming comfortable working with point-set topological spaces?

The key reason *we* will be studying simplicial sets is that we can use them to give one definition of $\infty$-groupoid; more precisely, we will define $\infty$-groupoids as a certain full subcategory of the category of simplicial sets.

At the moment, it is likely unclear why $\infty$-groupoids ought to embed in simplicial sets. In the next few chapters, we will develop the theory of simplicial sets and observe why they might be a good model of $\infty$-groupoids.

# Part I

# Simplicial sets

# Chapter 2

# Simplicial sets

## 2.1 Semisimplicial sets

In the previous chapter, we set the stage for simplicial sets: a combinatorial definition of shape, based on describing a set of pieces you need to build the shape, plus instructions on how to assemble those pieces.

Before defining simplicial sets, we give a related but simpler object: *semisimplicial sets*. Semisimplicial sets handle this idea of glueing pieces very well, but they turn out to have a minor technical defect which simplicial sets correct; it is hard to appreciate the definition of simplicial set until one sees the technical problem they solve, though, and hence we start with semisimplicial sets.

What is a semisimplicial set $X$? As explained above, we need to have pieces of various dimensions, together with instructions on how to assemble them into a shape.

The first piece of data in the definition is a set $X_0$, thought of as the set of *vertices* of the semisimplicial set.

The next step is to attach 1-dimensional pieces to these vertices; we take a set $X_1$, thought of as 1-dimensional edges, and two maps

$$d_1^0, d_1^1 : X_1 \to X_0.$$

These two maps encode the *endpoints* of our edge: the edge $e \in X_1$ should be attached in such a way that it goes from $d^1(e)$ to $d^0(e)$.

**Warning 2.1.1.** Note that $d_1^1(e)$ is the *source*, and $d_1^0(e)$ is the *target*; one can think that $d_1^i(e)$ means "the vertex *opposite* the $i^{\text{th}}$ vertex." It can seem strange to focus on the deleted vertex, but it will make more sense once we get to higher dimensional simplices.

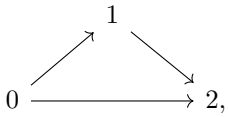
**Remark 2.1.2.** Observe that each edge has a *first* and a *last* vertex. In other words, edges are directed. It is more convenient to arbitrarily direct each edge than it would be to talk about some function $d : X_1 \to \mathrm{Sym}^2(X_0)$ that sent an edge to its pair of vertices, which is why we direct the edges. When we get to larger dimensional pieces, with more and more boundary components, picking labels for the boundaries becomes even more helpful.

We next attach 2-dimensional pieces. There are many choices one could make for what the 'basic' 2-dimensional piece is, but simplicial sets use the *triangle* as their basic 2-dimensional piece.

**Remark 2.1.3.** Any polygon can be built by glueing triangles together, and so the triangle is a pretty convenient basic 2-dimensional piece.

However, there are alternatives to the notion of simplicial sets, which use other shapes as your basic building block – just as you can approximate any 2-dimensional shape by triangles, you can also approximate any 2-dimensional shape by squares, and hence you might use squares as your basic 2-dimensional building blocks instead, leading to *cubical sets*. These notes will not mention cubical sets again.

A triangle has three sides; our glueing instructions will need to tell us which three edges in $X_1$ should become the sides of our triangle. It is convenient to label the vertices of the triangle as




so that we can refer to the vertex 1 or the edge 02 of our triangle.

Our glueing data consists of three maps

$$d_2^0, d_2^1, d_2^2 : X_2 \to X_1,$$

with $d_2^0$ sending a 2-simplex $\sigma$ to its 12 edge, $d_2^1$ sending $\sigma$ to its 02 edge, and $d_2^2$ sending $\sigma$ to its 01 edge.

**Remark 2.1.4.** It's easiest to remember this by thinking that $d_2^i(\sigma)$ is the edge of $\sigma$ *opposite* from vertex $i$.

We also require certain relations between these three maps: because the 01 edge and the 02 edge share a common source, for example, we demand

$$d_1^1(d_2^2(\sigma)) = d_1^1(d_2^1(\sigma)).$$

We can continue in this way, letting $X_n$ denote a set of *$n$-simplices*. One can think of an $n$-simplex geometrically as being the subset of $[0,1]^{n+1}$ described by the equation $x_0 + x_1 + \cdots + x_n = 1$. This is a polytope with $n+1$ vertices, which we label $0, 1, ..., n$. We then have boundary maps

$$d_n^0, d_n^1, ..., d_n^n : X_n \to X_{n-1},$$

with $d_n^i(\sigma)$ being the face of $\sigma$ opposite to vertex $i$.

**Ideation 2.1.5** (Relations between boundaries)**.** As described above, a semisimplicial set is a collection of sets, with certain functions between them, obeying certain relations involving function composition. It's easiest to encode this by hoping we can define a category $\mathcal{C}$ which has one object $[n]$ for every natural number $n \in \{0, 1, ...\}$, in such a way that semisimplicial sets are the same thing as functors

$$X : \mathcal{C} \to \text{Set}\,.$$

The sets $X_n$ are just the values $X([n])$ of our functor, and the boundary maps

$$d_n^i : X_n \to X_{n-1}$$

come from applying $X$ to maps $d^i_n : [n] \to [n-1]$.

How should we define the morphisms in $\mathcal{C}$ to get correctly behaving boundary maps? The idea is that a face of the standard $n$-simplex is determined by which vertices it contains. So, if we think about $[n] \in \mathcal{C}$ as being the finite set $\{0, 1, ..., n\}$ of all the vertices of an $n$-simplex, then the maps $d^i_n : [n] \to [n-1]$ can correspond to the $n+1$ different *subsets* of $[n]$ of size $n$.

This might suggest defining $\mathcal{C}$ to be the *opposite* of the category $\mathrm{FinSet}_{\mathrm{inj}}$; that is, the category whose objects are finite sets, and where morphisms are *injective* maps. However, this category is a bit too large: in $\mathrm{FinSet}_{\mathrm{inj}}$, the set $\{0, 1, ..., n\}$ has a huge automorphism group. To fix this, we will rigidify the situation.

**Definition 2.1.6.** We let $\Delta_{\mathrm{inj}}$ be the category whose objects are nonempty, finite, totally ordered sets, and whose morphisms are *injective*, order-preserving functions.

**Remark 2.1.7.** Every object of $\Delta_{\mathrm{inj}}$ is isomorphic to a set $[n] = \{0, 1, ..., n\}$ with order relation

$$0 < 1 < \cdots < n,$$

for some $n \geq 0$.

Thus requiring our objects be totally ordered sets is equivalent to labelling each vertex, adding the rigidity we desired in Ideation 2.1.5.

**Definition 2.1.8.** A *semisimplicial set* is a functor

$$X : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathrm{Set}\,.$$

To further justify this definition, we give the following lemma describing the combinatorics of $\Delta_{\mathrm{inj}}$.

**Lemma 2.1.9.** *For $0 \leq i \leq n+1$, let*

$$d^i_{n+1} : [n] \to [n+1]$$

*be the unique monotone injective map which omits $i$ from its range.*

*Then every morphism in $\Delta_{\mathrm{inj}}$ can be obtained as a composition of the morphisms $d^i_{n+1}$ (where empty composition gives identity by convention).*

*Proof.* If $f : [n] \to [n+k]$ is a monotone injection, for $k > 0$, let the range of $f$ omit the values $a_1 < a_2 < \cdots < a_k$.

As $a_i \leq n + i$, it always makes sense to form $d^{a_i}_{n+i}$. We then have

$$f = d^{a_k}_{n+k} \circ \cdots \circ d^{a_2}_{n+2} \circ d^{a_1}_{n+1},$$

our desired factorization. □

**Notation 2.1.10.** Let $X_\bullet$ be a semisimplicial set. We will often write

$$d^i_{n+1} : X_{n+1} \to X_n$$

as an abbreviation for $X_\bullet(d^i_{n+1})$, where $d^i_{n+1}$ is as in Lemma 2.1.9.

**Remark 2.1.11.** The monotone injection $d^i_{n+1} : [n] \to [n+1]$ omits $i$ from its range; in other words, $d^i_{n+1}$ encodes the subset of $[n+1]$ obtained by deleting $i$. This fits with Remark 2.1.4 from earlier: $d^i_{n+1}$ picks out the face *opposite to* vertex $i$ (or equivalently, the unique face not containing vertex $i$).

## 2.2 Simplicial sets

One annoyance of semisimplicial sets is that products behave quite poorly.

**Ideation 2.2.1.** To represent the unit interval $[0,1]$ as a semisimplicial set, it seems most natural to take

$$X_0 = \{0,1\},$$
$$X_1 = \{e\},$$
$$X_2 = X_3 = \cdots = \emptyset,$$

with $d^0(e) = 1, d^1(e) = 0$.

In other words: the interval is formed by taking two points, and attaching an edge connecting them.

Next, let's try forming the unit square $[0,1]^2$ as a semisimplicial set. While one could do this directly, it seems better to hope that we could form this as some kind of product $X_\bullet \times X_\bullet$ of the unit interval with itself.

Fortunately for us, the category of functors $\Delta_{\text{inj}}^{\text{op}} \to \text{Set}$ has all limits, and those limits are easy to compute: evaluate pointwise. Hence we might hope that $[0,1]^2$ is represented by

$$Y_0 = \{(0,0),(0,1),(1,0),(1,1)\},$$
$$Y_1 = \{(e,e)\},$$
$$Y_2 = Y_3 = \cdots = \emptyset.$$

Unfortunately, these glueing instructions definitely do not describe a square: we have four points, and then attach an edge going from $(0,0)$ to $(1,1)$. This is a line with two extra points floating off beside it, not a square!

To remedy this, let's think about how we might create a simplicial square. The easiest way seems to be to glue two triangles together along one common edge, as depicted in Figure 2.1.



Figure 2.1: A simplicial square.

We want this simplicial square to be the product of the unit interval with itself. But note that something funny happens: when we project the $(0,0) \to (1,0)$ edge onto the $(0,0) \to (0,1)$ side, we just get a *point*. This suggests that the image of this edge under one of the two projections $\Delta_1 \times \Delta_1 \to \Delta_1$ should just be a single point.

To accommodate this, we might try adding *length-zero 'degenerate' edges* to our semisimplicial sets. That is, consider the semisimplicial set

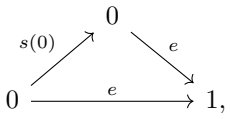
$$X_0' = \{0,1\}, X_1' = \{s(0), s(1), e\}, X_2' = X_3' = \cdots = \emptyset,$$

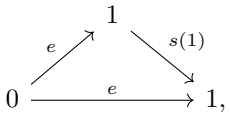
where $d^0(s(i)) = d^1(s(i)) = i$. Then $X_1' \times X_1'$ is a *nine* element set, consisting of:

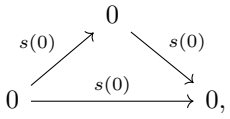
1. the edge $(e, e)$, which corresponds to the diagonal of our square (because it is something whose projection onto either axis is $e$),
2. the edges $(s(0), e), (s(1), e)$, which correspond to the two vertical sides of our square (because their projection to the $x$-axis is a single point, and their projection to the $y$-axis is $e$),
3. the edges $(e, s(0)), (e, s(1))$, which correspond to the two horizontal sides of our square,
4. the four edges $(s(0), s(0)), (s(0), s(1)), (s(1), s(0)), (s(1), s(1))$, which are the *degenerate edges* of the square, in the same way that $s(0), s(1)$ were the degenerate edges of $X'_\bullet$.

While the product $X'_1 \times X'_1$ looks right to capture the simplicial square, unfortunately $X'_2 \times X'_2$ is still empty.

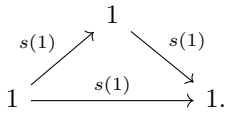
As degenerate edges fixed the edges of our product, to fix the 2-simplices of the product, let's try adding degenerate 2-simplices. But how many? To get two non-degenerate 2-simplices in the product, we will need at least two degenerate 2-simplices in $[0, 1]$. To accomplish this, we just make a 2-simplex where at least one side is a degenerate edge of length zero; the four resulting 2-simplices are




and

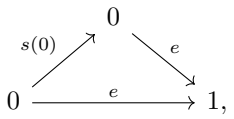



Unfortunately, even after adding these degenerate 2-simplices, we still have a problem: when viewing a semisimplicial set as glueing instructions, how do we distinguish between an edge of length zero like $s(0)$ from an edge which is supposed to be a loop around 0? For example, we might want to model a simplicial circle with one vertex and one loop both of whose endpoints are attached at that vertex.

To handle this, we add in explicit *degeneracy morphisms* $X_i \to X_{i+1}$ which encode which simplices are degenerate. But how many should we add, and what relations should they obey? To get a feel for degeneracy morphisms, we mention three kinds of degeneracies we've already seen:

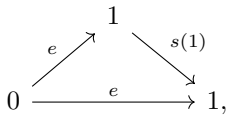
1. $s_0^0 : X_0 \to X_1$ sends each vertex to some length zero degenerate loop at that vertex;

2. $s_1^0 : X_1 \to X_2$ sends an edge $e$ (say from $0 := d^1(e)$ to $1 := d^0(e)$) to a 2-simplex with boundary




which we think of as the 2-simplex obtained by "repeating" the vertex 0, using a degenerate edge;

3. $s_1^1 : X_1 \to X_2$ sends $e$ to a 2-simplex




which we think of as being obtained by repeating the vertex 1 using a degenerate edge.

Recall that semisimplicial sets are functors

$$\Delta_{\text{inj}}^{\text{op}} \to \text{Set}\,.$$

The injectivity is because we only want to have maps $X_n \to X_{n-1}$ from simplices to *smaller* simplices, encoding boundaries; boundaries were found by 'forgetting' a vertex and deleting it from our simplex (see Remark 2.1.11). To accommodate these degeneracies, the simplest thing to do is to enlarge $\Delta_{\text{inj}}$ by allowing functions

$$[n] \to [n-1]$$

which, instead of *deleting* an element from a set to get a subset, *repeat* an element, like with the function

$$\{0,1,2\} \to \{0,1\},$$
$$0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1,$$

where we *repeat* the element 0 of $\{0,1\}$.

If we want these 'repeating vertex' maps, then all monotone functions come along for the ride.

**Notation 2.2.2.** For $0 \leq i \leq n$, we write $s_n^i : [n+1] \to [n]$ for the monotone surjection having values

$$s_n^i(0) = 0, s_n^i(1) = 1, ..., s_n^i(i) = i, s_n^i(i+1) = i, s_n^i(i+2) = i+1, ..., s_n^i(n+1) = n.$$

**Lemma 2.2.3.** *Every* monotone *function is either the identity, or can be written as a composition of the functions $s_n^i$ and $d_n^i$. Moreover, every non-identity surjective function can be written only using the $s_n^i$.*

*Proof.* Let $f : [n] \to [m]$ be monotone. Factor $f$ as

$$[n] \xrightarrow{\pi} \text{im}(f) \xrightarrow{i} [m],$$

where

1. $i : \mathrm{im}(f) \hookrightarrow [m]$ is the image of $f$,
2. $\pi : [n] \to \mathrm{im}(f)$ is given by $\pi(i) = f(i)$, using that $f(i) \in [m]$ always belongs to the subset $\mathrm{im}(f)$.

Thus it suffices to prove the latter part of our lemma, since every injection can be obtained using the $d$'s by Lemma 2.1.9.

And if

$$f : [n] \to [n-k]$$

is surjective, for $k > 0$, then we prove $f$ is a composition of the maps $s_i^j$ by induction on $k$. When $k = 1$, this is obvious as every monotone surjection $[n] \to [n-1]$ is one of the $s_{n-1}^i$. For the induction step, let $a \in [n]$ be the least element such that there exists some $b \in [n]$ with $f(a) = f(b)$ but $b \neq a$.

Choose now $b$ the *largest* element of $[n]$ so that $f(a) = f(b)$. Put $\alpha := f(a)$. By monotonicity, it's easy to see that $f(a) = f(a+1) = \cdots = f(b) = \alpha$. Put

$$g : [n] \to [n-k] \cup \{\alpha + 0.5\}$$

(ordering the codomain in the obvious way, with $\alpha < \alpha + 0.5 < \alpha + 1$) the function

$$g(i) = \begin{cases} f(i) & i \leq a, \\ \alpha + 0.5 & a+1 \leq i \leq b, \\ f(i) & i > b. \end{cases}$$

This function $g$ is monotone, as if $i > b$, then

$$f(i) \geq f(b) = \alpha,$$

but also $f(i) \neq f(b)$ by maximality of $b$, and hence $f(i) \geq \alpha + 1$.

We can write

$$f = \pi \circ g,$$

where

$$\pi : [n-k] \cup \{\alpha + 0.5\} \to [n-k]$$

is the surjection sending $\alpha + 0.5$ to $\alpha$ and otherwise acting as the identity. By the base case (for $\pi$) and the induction hypothesis (for $g$), we conclude that $f$ is a composition of the degeneracies, and so we conclude. □

Inspired by Ideation 2.2.1 and Lemma 2.2.3, we make the following definition.

**Definition 2.2.4.** A *simplicial object* in a category $\mathcal{A}$ is a functor

$$X : \Delta^{\mathrm{op}} \to \mathcal{A},$$

where $\Delta$ is the category of all non-empty, totally ordered, finite sets, with monotone functions as morphisms.

We write $\mathrm{Simp}(\mathcal{A})$ to denote the category of simplicial objects of $\mathcal{A}$, and sSet to denote the category of simplicial objects in Set .

**Definition 2.2.5.** Let $X$ be a simplicial set. An $n$-simplex $\sigma \in X_n$ is said to be *degenerate* if it lies in the image of some degeneracy map $s^i_{n-1} : X_{n-1} \to X_n$. If $\sigma$ is not degenerate, we say it is *non-degenerate*.

**Remark 2.2.6.** In particular, every 0-simplex is non-degenerate.

**Remark 2.2.7.** Recall that a groupoid can be defined as a pair of sets $A, X$ together with maps

$$s, t : A \to X,$$

$$\mathrm{id} : X \to A,$$

together with a composition and an inverse. The source and target maps are exactly like the two face maps $X_1 \to X_0$ in a simplicial set, and the identity map is exactly like the degeneracy $X_0 \to X_1$. This hints at the connection between groupoids and simplicial sets.

We can give a combinatorial description of the category $\Delta$.

**Lemma 2.2.8.** *Monotone functions $[n] \to [m]$ are in bijection with lists $a_0, a_1, ..., a_n$ where $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n \leq m$.*

*The set of functions in $\Delta$ is generated under composition by the identity maps, the maps $d^i_n$, and the maps $s^i_n$.*

*Moreover, a functor $X : \Delta^{\mathrm{op}} \to \mathcal{A}$ is completely determined by a system of objects $X_n$, and maps $d_i : X_n \to X_{n-1}, s_i : X_n \to X_{n+1}$ obeying relations $d_i d_j = d_{j-1} d_i$ and $d_i s_j = s_{j-1} d_i$ for $i < j$, and $s_i s_j = s_{j+1} s_i$ for $i \leq j$, and $d_j s_j = \mathrm{id}, d_{j+1} s_j = \mathrm{id}$, and $d_i s_j = s_j d_{i-1}$ for $i > j + 1$.*

*Proof.* This is essentially Lemma 2.2.3, but with a little more work for relations; we leave it to the reader as an exercise. □

While simplicial sets make products better behaved, they seem to defeat the point of semisimplicial sets: a semisimplicial set can be defined with a small amount of geometrically intuitive information, whereas simplicial sets require one to use these large numbers of degenerate simplices – and while degenerate simplices do get more intuitive over time, they're definitely a bit more annoying than the simplices of a semisimplicial set.

Fortunately, one often gets the best of both worlds: the language of simplicial sets is robust enough that in practice one doesn't need to think very hard about degenerate simplices, but the machinery of simplicial sets is powerful enough that everything works well.

To start getting a handle on degeneracies, we prove the following basic result.

**Proposition 2.2.9.** *Let $X$ be a simplicial set. Then every $n$-simplex $\sigma$ can be written* uniquely *as $\sigma = \alpha^*(\tau)$, where $\tau$ is a non-degenerate $k$-simplex for $k \leq n$ and $\alpha : [n] \to [k]$ is a uniquely determined monotone surjection.*

*Proof.* We induct on $n$. The base case of $n = 0$ is obvious.

For the induction step, we first get existence. If $\sigma$ is non-degenerate, then $\sigma = \mathrm{id}^*(\sigma)$, and uniqueness is immediate.

Assume now that $\sigma$ is degenerate. Then we may write

$$\sigma = s^i_{n-1}(\sigma')$$

for some $i$ and some $\sigma'$, just by definition of degeneracy. Applying the induction hypothesis, we can write $\sigma' = \beta^*(\tau)$, and hence $\sigma = \alpha^*(\tau)$ for $\alpha = \beta \circ s^i_{n-1}$. This gives us existence.

To see uniqueness, assume we could write

$$\sigma = \alpha^*(\tau) = (\alpha')^*(\tau')$$

with $\tau, \tau'$ both non-degenerate simplices of dimensions $k, k' \leq n$. As we assumed $\sigma$ was degenerate, we in fact have $k, k' < n$, as it cannot be the case that $\sigma = \tau$ or $\sigma = \tau'$.

As $\alpha$ is a surjection, we can choose a monotone section $s : [k] \hookrightarrow [n]$ of $\alpha$. Thus $s^*(\sigma) = \tau$. Hence

$$(\alpha' s)^*(\tau') = \tau.$$

The composite map $\alpha' s : [k] \to [k']$ is some monotone map which takes a non-degenerate simplex to a non-degenerate simplex; it therefore must be injective, and in particular $k \leq k'$. But the same argument shows $k' \leq k$, and so $k = k'$.

Thus $\alpha' s$ is actually the identity function; hence $(\alpha' s)^*(\tau') = \tau'$, so that $\tau = \tau'$.

Finally, we must show $\alpha = \alpha'$. As $\alpha' s = \alpha s = \mathrm{id}$, we know $\alpha = \alpha'$ on the image of $s$; but by varying the choice of $s$, we can cover all of $[n]$, and hence get $\alpha = \alpha'$. □

## 2.3 Some basic examples of simplicial sets

### 2.3.1 Simplices

**Definition 2.3.1.** We define the *$n$-simplex* $\Delta_n$ as the simplicial set corresponding to the presheaf $\mathrm{Mor}(-, [n])$.

**Remark 2.3.2.** An $i$-simplex of $\Delta_n$ is just a list of integers $a_0, ..., a_i$ such that

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_i \leq n.$$

Note that a non-degenerate $i$-simplex is therefore *uniquely determined by its vertices*; in a general simplicial set, there can be many simplices with the same vertices (for example, we could model the simplicial circle by taking two vertices $a$ and $b$, and adding two parallel edges $e_1, e_2$ going from $a$ to $b$; then $e_1 \neq e_2$, but $e_1, e_2$ have the same vertices).

The non-degenerate $i$-simplices of $\Delta_n$ are just those lists which obey

$$a_0 < a_1 < \cdots < a_i.$$

In particular, non-degenerate $i$-simplices of $\Delta_n$ are in bijection with $i+1$ element subsets of $[n]$.

### 2.3.2 The nerve of a poset

Related to Definition 2.3.1, not only do all totally ordered sets give rise to simplicial sets, but even all partially ordered sets do.

**Construction 2.3.3** (Nerve of a poset)**.** Let $P$ be a poset. We define its nerve $N(P) \in \mathrm{sSet}$ to be the simplicial set where

1. $N(P)_n$ is the set of all $(n+1)$-tuples $(p_0, ..., p_n)$ with $p_0 \leq p_1 \leq \cdots \leq p_n$; that is, $N(P)_n$ is the set of all morphisms of posets $[n] \to P$;
2. for any monotone function $f : [n] \to [m]$, the corresponding map $N(P)_m \to N(P)_n$ is

$$(p_0, ..., p_m) \mapsto (p_{f(0)}, ..., p_{f(n)}).$$

### 2.3.3 0-dimensional simplicial sets

**Example 2.3.4** (Constant simplicial objects)**.** Let $\mathcal{A}$ be a category. We define a functor

$$\mathcal{A} \to \mathrm{Simp}(\mathcal{A})$$

sending $A \in \mathcal{A}$ to the simplicial object $X_\bullet$ with

$$X_0 = A, X_1 = A, ...,$$

with all boundary and degeneracy maps being the identity.

We call this the *constant simplicial object*, and denote it by $\underline{A}$. The functor

$$A \mapsto \underline{A}$$

is easily seen to be fully faithful.

**Remark 2.3.5.** We think of a constant simplicial set $\underline{A}$ as being a discrete, 0-dimensional space, with set of vertices $A$.

### 2.3.4 Simplicial subsets

**Definition 2.3.6.** A *simplicial subset* of a simplicial set $X$ is a simplicial set $Y$ together with a map $Y \to X$ which is injective on $n$-simplices for all $n$.

**Notation 2.3.7.** If $Y \to X$ is a simplicial subset, we will often refer to it as a *monomorphism* (or sometimes just as *monic*), or an *inclusion* of simplicial sets. Many sources refer to inclusions as *cofibrations*, but we will not use this term.

**Construction 2.3.8** (The subsimplicial set spanned by simplices)**.** Let $X$ be a simplicial set, and $\{S_i \subseteq X_i\}_{i \geq 0}$ a collection of sets of simplices of $X$ of various dimensions.

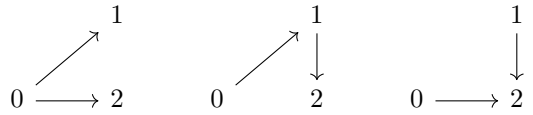
The *subsimplicial set of $X$ spanned by $S$* is the simplicial subset $Y$ of $X$ defined as the intersection of all subsimplicial sets $K$ of $X$ for which $S_i \subseteq K_i$ for all $i$. This collection of subsimplicial sets $K$ is nonempty (as it always contains $X$), and an arbitrary intersection of subsimplicial sets is obviously a subsimplicial set.

As an example of subsimplicial sets, let's discuss *horns*.

**Definition 2.3.9.** Let $n \geq 1$ and $0 \leq i \leq n$. We define the $(n, i)$-horn $\Lambda^i_n$ to be the subsimplicial set of $\Delta_n$ spanned by all the non-degenerate $(n-1)$-simplices *except* for the one opposite to vertex $i$.

**Example 2.3.10.** We have $\Lambda^0_1 = \{0\}, \Lambda^1_1 = \{1\}$.

**Example 2.3.11.** The horns $\Lambda^0_2, \Lambda^1_2$, and $\Lambda^2_2$, drawn from left to right, look as follows.




The proof of the following proposition is instructive.

**Proposition 2.3.12.** *Let $X$ be a simplicial set. Then*

$$\mathrm{Mor}_{\mathrm{sSet}}(\Lambda^1_2, X) \simeq \{(f,g) \in X_1^2 \mid d^0(f) = d^1(g)\},$$

*and this bijection can be constructed such that it is functorial in $X$.*

*In particular, $\Lambda^1_2$ is the pushout*

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{0} & \Delta_1 \\ \downarrow{\scriptstyle 1} & & \\ \Delta_1. & & \end{array}$$

**Remark 2.3.13.** Recall that $d^0(f)$ is the target of $f$, and $d^1(g)$ is the source of $g$.

*Proof.* It suffices to prove the pushout statement; set $A = \Delta_1 \coprod_{\mathrm{pt}} \Delta_1$. There is a natural map

$$i : A \to \Delta_2$$

sending the first $\Delta_1$ to 01 and the second $\Delta_1$ to 12.

As colimits in sSet can be computed pointwise, we have

$$A_k = (\Delta_1)_k \coprod_{\mathrm{pt}_k} (\Delta_1)_k.$$

In particular,

$$i_k : A_k \to (\Delta_2)_k$$

is injective: a $k$-simplex coming from the first $(\Delta_1)_k$ will be sent to a monotone list in $\{0,1,2\}$ consisting of zeroes and ones, and a $k$-simplex coming from the second $(\Delta_1)_k$ will be sent to a monotone list consisting of ones and twos; thus the only collision comes from the list of all ones in the first factor being sent to the same spot as the list of all zeroes in the second factor, but the coproduct in $A_k$ identifies those two simplices.

Thus $A$ is a simplicial subset of $\Delta_2$ containing the 01 and 12 edges. As all simplices of $A$ are produced by taking faces or degeneracies of those two edges, it is clear that $A$ is the smallest simplicial subset of $\Delta_2$ containing these edges, and hence must be the horn $\Lambda^1_2$, as desired. □

## 2.4 Filling in a simplicial object

In our march towards making simpler ways to construct simplicial sets, we begin in this section by creating a machine which automatically adds in all the high-dimensional degenerate simplices. That is, a simplicial set modelling a 3-dimensional shape, say, will need degenerate simplices in dimension 4, 5, 6, etc., and it can be annoying to have to specify those after you've already specified the 3-dimensional ones.

This machine does not totally solve our problems, because it still requires us to specify the degenerate simplices of dimensions 1, 2, and 3, but it definitely helps.

**Definition 2.4.1.** An *$n$-truncated simplicial object* of a category $\mathcal{A}$ is a functor $\Delta^{\mathrm{op}}_{\leq n} \to \mathcal{A}$, where $\Delta_{\leq n}$ is the full subcategory of $\Delta$ whose objects are $[0], [1], ..., [n]$. We let $\mathrm{Simp}_n(\mathcal{A})$ denote the category of $n$-truncated simplicial objects of $\mathcal{A}$.

**Definition 2.4.2.** The *skeleton functor* is the functor

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Simp}_n(\mathcal{A})$$

given by restriction along $\Delta^{\mathrm{op}}_{\leq n} \hookrightarrow \Delta^{\mathrm{op}}$.

**Remark 2.4.3.** Some sources call this functor truncation instead.

The question of filling in a simplicial object with degenerate objects is then the question of finding a left adjoint to $\mathrm{sk}_n$ . Fortunately, such a left adjoint exists under very mild assumptions on the category $\mathcal{A}$.

**Theorem 2.4.4.** *Assume $\mathcal{A}$ has all finite colimits. Then the skeleton functor $\mathrm{sk}_n$ has a fully faithful left adjoint $i_{n!} : \mathrm{Simp}_n(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$.*

*In fact, for $X \in \mathrm{Simp}_n(\mathcal{A})$, we have*

$$(i_{n!}X)_k = \mathrm{colim}_{[k]\to[j],j\leq n} X_j.$$

**Remark 2.4.5.** To be a bit more precise, we take the colimit over the coslice category $([k]/\Delta)^{\mathrm{op}}_{\leq n}$. In some sense, this left adjoint is just freely adjoining whichever degenerate objects in dimension $k$ have to come from the cells in dimensions bounded by $n$.

*Proof.* Define

$$\alpha : \mathrm{Mor}(X, \mathrm{sk}_n(Y)) \to \mathrm{Mor}(i_{n!}X, Y)$$

as the map sending $f : X \to \mathrm{sk}_n(Y)$ to the morphism

$$\alpha(f) : i_{n!}X \to Y$$

which, on $k$-simplices, is given by the map

$$\mathrm{colim}_{[k]\to[j],j\leq n} X_j \to Y_k$$

whose restriction to the $\varphi : [k] \to [j]$ factor is

$$X_j \xrightarrow{f_j} Y_j \xrightarrow{\varphi^*} Y_k.$$

This construction is functorial in $X, Y$, so it suffices to prove that $\alpha$ is a bijection to conclude. As

$$X_k \to \mathrm{colim}_{[k]\to[j],j\leq n} X_j$$

is an isomorphism when $k \leq n$ (use that $\mathrm{id}_{[k]}$ is a final object of our diagram category), it is easy to see that $\alpha$ is injective.

For surjectivity, let

$$g : i_{n!}X \to Y$$

be a morphism of simplicial objects. Define

$$f : X \to \mathrm{sk}_n(Y)$$

by $f_k := g_k$ for all $k \leq n$. The compatibility of $g$ with simplicial set maps easily implies $g = \alpha(f)$.

For the second part, we must show that the unit map

$$\eta : \mathrm{id} \to \mathrm{sk}_n \, i_{n!}$$

is an isomorphism. But, for $X \in \mathrm{Simp}_n(\mathcal{A})$, the unit

$$\eta_X : X \to \mathrm{sk}_n(i_{n!}X)$$

is given on $k$-simplices by the isomorphism

$$X_k \to \mathrm{colim}_{[k]\to[j], j\leq n} \, X_j,$$

and so we conclude. □

**Warning 2.4.6.** As $i_{n!}$ is fully faithful, we will often abuse notation and identify $\mathrm{Simp}_n(\mathcal{A})$ with its essential image. In particular, we often consider $\mathrm{sk}_n(Y)$ as a simplicial object, instead of a truncated simplicial object.

The functor $\mathrm{sk}_n$ commutes with all colimits, because colimits are computed pointwise. In particular, we should expect it is also a *left* adjoint, and indeed it is.

**Theorem 2.4.7.** *Assume $\mathcal{A}$ has all finite limits. The functor $\mathrm{sk}_n$ is left adjoint to a functor*

$$\mathrm{cosk}_n : \mathrm{Simp}_n(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$$

*given by*

$$\mathrm{cosk}_n(X)_k := \lim_{[j]\to[k], j\leq n} X_j.$$

*Proof.* Let $X \in \mathrm{Simp}_n(\mathcal{A}), Y \in \mathrm{Simp}(\mathcal{A})$. We construct a functorial bijection

$$\theta : \mathrm{Mor}(\mathrm{sk}_n(Y), X) \to \mathrm{Mor}(Y, \mathrm{cosk}_n(X))$$

as follows.

We define

$$\theta(f) : Y \to \mathrm{cosk}_n(X)$$

the map

$$\theta(f)_k : Y_k \to \lim_{[j]\to[k], j\leq n} X_j$$

induced by the maps $Y_k \xrightarrow{\varphi^*} Y_j \xrightarrow{f_j} X_j$.

It's easy to see that $\theta$ is injective, because $\mathrm{cosk}_n(X)_j = X_j$ if $j \leq n$. For surjectivity, take a map $g : Y \to \mathrm{cosk}_n(X)$. Define

$$f : \mathrm{sk}_n(Y) \to X$$

the map $f_k := g_k$, using that $\mathrm{sk}_n(Y)_k = Y_k$. We must prove that, for all $k > n$, we have $\theta(f)_k = g_k$.

And indeed, because $g$ is a morphism of simplicial sets, for any $\varphi : [j] \to [k]$ with $j \leq n$, we find $\varphi^* \circ g_k = g_j \circ \varphi^* = f_j \circ \varphi^*$; but $\theta(f)_k$ is the map uniquely characterized by those formulas, so we conclude. □

### 2.4.1 Dimension

The functor

$$i_{n!} : \mathrm{sSet}_{\leq n} \to \mathrm{sSet}$$

studied above is fully faithful. Whenever one has a fully faithful functor, it is interesting to study its image.

**Definition 2.4.8.** Let $S$ be a simplicial set. We say that $S$ *has dimension* $\leq n$ if $S$ has no non-degenerate simplices of dimension $d > n$. We say that $S$ *has dimension* $n$ if $S$ has dimension $\leq n$ and also has a non-degenerate $n$-simplex.

**Theorem 2.4.9.** *A simplicial set $S$ is of dimension $\leq n$ if and only if it belongs to the essential image of $i_{n!}$.*

*Proof.* First assume $\dim(S) \leq n$. To prove that $S$ belongs to the essential image of $i_{n!}$, it suffices to prove that the counit map

$$\epsilon_S : i_{n!}(\mathrm{sk}_n S) \to S$$

is an isomorphism.

We have

$$i_{n!}\,\mathrm{sk}_n(S)_k = \mathrm{colim}_{j\leq n,[k]\to[j]}\, S_j.$$

We wish to prove that

$$\epsilon_k : \mathrm{colim}_{j\leq n,[k]\to[j]}\, S_j \to S_k$$

is a bijection for all $k$.

When $k \leq n$, this is easy: $\mathrm{id}_{[k]}$ is a final object in our diagram category, so the colimit is just $S_k$ and $\epsilon_k = \mathrm{id}_{S_k}$.

When $k > n$, things are a little trickier. For injectivity, it suffices to prove that if $\varphi : [k] \to [j], \varphi' : [k] \to [j']$ are monotone maps with $j, j' \leq n$, and $\tau \in S_j, \tau' \in S_{j'}$ two simplices so that

$$\varphi^*(\tau) = (\varphi')^*(\tau'),$$

then $(\tau, \varphi), (\tau', \varphi')$ are equal in the colimit.

Factoring $\varphi = i \circ \pi$ as the composition of an injection and a surjection, we can replace $(\tau, \varphi)$ with $(i^*\tau, \pi)$ in the colimit to assume $\varphi$ is surjective. Similarly, we may assume $\varphi'$ is a surjection.

Write

$$\tau = \alpha^*(\sigma),$$
$$\tau' = (\alpha')^*(\sigma'),$$

with $\sigma, \sigma'$ both non-degenerate, and

$$\alpha : [j] \to [\ell], \alpha' : [j'] \to [\ell']$$

two monotone surjections. In particular, $\ell, \ell' \leq n$.

Thus

$$\varphi^*(\alpha^*(\sigma)) = (\varphi')^*((\alpha')^*(\sigma')),$$

so the uniqueness in Proposition 2.2.9 (recalling that $\varphi, \varphi'$ are surjections) implies $\ell = \ell'$,

$$\alpha \circ \varphi = \alpha' \circ \varphi',$$

and $\sigma = \sigma'$.

Thus, in the final colimit, both $(\tau, \varphi)$ and $(\tau', \varphi')$ get identified with $(\sigma, \alpha\varphi)$, and so we conclude injectivity.

Surjectivity is immediate from Proposition 2.2.9, so we conclude the desired bijectivity. Hence $\epsilon_S$ is an isomorphism.

Conversely, $i_{n!}X$ never has any non-degenerate simplices of dimension exceeding $n$, as is seen immediately from its construction. □

### 2.4.2 The skeletal filtration

We can write every simplicial set as a colimit of finite-dimensional ones.

**Construction 2.4.10** (Skeletal filtration)**.** Let $X_\bullet$ be a simplicial set. Define

$$X(n) := i_{n!}\,\mathrm{sk}_n(X_\bullet) \in \mathrm{sSet}$$

the $n^{\text{th}}$ skeleton of $X$. The *skeletal filtration* of $X_\bullet$ is the series of maps

$$\emptyset =: X(-1) \to X(0) \to X(1) \to \cdots$$

we now construct.

By the $(i_{(n-1)!}, \mathrm{sk}_{n-1})$ adjunction, we have

$$\mathrm{Mor}(X(n-1), X(n)) = \mathrm{Mor}(\mathrm{sk}_{n-1}(X), \mathrm{sk}_{n-1}\, i_{n!}\, \mathrm{sk}_n(X)) = \mathrm{Mor}(\mathrm{sk}_{n-1}(X), \mathrm{sk}_{n-1}(X)).$$

Let $X(n-1) \to X(n)$ be the adjunct of the identity map $\mathrm{sk}_{n-1}(X) \to \mathrm{sk}_{n-1}(X)$.

**Notation 2.4.11.** We sometimes abuse notation and write just $X(n) = \mathrm{sk}_n(X_\bullet)$, using that $i_{n!}$ is fully faithful and hence omitting it tends to not be bad.

**Proposition 2.4.12.** *For a simplicial set $X$, the map*

$$X(n) \to X$$

*is an inclusion; $X(n)$ is the subsimplicial set given by*

$$X(n)_k = X_k$$

*for $k \leq n$, and*

$$X(n)_k = \{\sigma \in X_k \mid \sigma \text{ is a degeneracy of a } j\text{-simplex, } j \leq n\}$$

*for $k > n$.*

**Remark 2.4.13.** By '$\sigma$ is a degeneracy of a $j$-simplex,' we mean there is some non-degenerate $j$-simplex $\tau \in X_j$ such that $\sigma = \alpha^*(\tau)$ for some surjective and non-identity monotone map $\alpha$.

*Proof.* By definition,

$$X(n)_k = \mathrm{colim}_{[k]\to[j], j\leq n} X_j.$$

If $k \leq n$, then $\mathrm{id}_{[k]}$ is final in the diagram category and hence $X(n)_k = X_k$.

Fix $k > n$, and set

$$S_k := \{\sigma \in X_k \mid \sigma \text{ is a degeneracy of a } j\text{-simplex, } j \leq n\}.$$

If $k > n$, then the map

$$X(n)_k \to X_k$$

of course has image equal to $S_k$.

To see injectivity, suppose we have monotone surjections $\varphi : [k] \to [j], \varphi' : [k] \to [j']$ with $j, j' \leq n$, and $\tau \in X_j, \tau' \in X_{j'}$ such that

$$\varphi^*(\tau) = (\varphi')^*(\tau').$$

Write this common simplex as

$$\alpha^*(\sigma)$$

for $\sigma$ non-degenerate, using Proposition 2.2.9. Applying Proposition 2.2.9 to $\tau, \tau'$, and using the uniqueness, we find that $(\alpha, \sigma)$ dominates $(\varphi, \tau)$ and $(\varphi', \tau')$ in the colimit of sets used to calculate $X(n)_k$, so that $(\varphi, \tau) = (\varphi', \tau')$ in $X(n)_k$.

To get injectivity in general, factor monotone maps as surjections followed by injections. □

**Proposition 2.4.14.** *For a simplicial set $X$, the natural map*

$$\operatorname{colim}_n X(n) \to X$$

*is an isomorphism.*

*Proof.* It suffices to check it is an isomorphism on $k$-skeletons for all $k$. But for any $n \geq k$, the map $X(n) \to X(n+1)$ is an isomorphism on $k$-skeletons, and so

$$\operatorname{sk}_k (\operatorname{colim}_n X(n)) = \operatorname{sk}_k X(k) = \operatorname{sk}_k X,$$

as $\operatorname{sk}_k$ is a left adjoint and hence preserves colimits. □

In the skeletal filtration, we saw an infinite chain of maps

$$X(0) \to X(1) \to X(2) \to \cdots$$

which has colimit $X$. This inspires the following notation.

**Definition 2.4.15** ($\omega$-composition)**.** Let

$$X_0 \to X_1 \to X_2 \to \cdots$$

be an infinite chain of morphisms. The *$\omega$-composition* of them is the resulting map

$$X_0 \to \operatorname{colim}_n X_n.$$

### 2.4.3 Glueing simplices

A simplicial set, informally, is a list of instructions on how to glue together simplices to get a shape. This suggests that every simplicial set should be obtained as an appropriate colimit of the standard $n$-simplices; in Proposition 2.3.12, we saw an example of this for $\Lambda^1_2$.

When glueing on the new $n$-simplices, we attach along their boundaries; we give some notation for this.

**Definition 2.4.16.** For $n \geq 0$, we define $\partial\Delta_n$ the subsimplicial set of $\Delta_n$ spanned by all the $(n-1)$-simplices of $\Delta_n$.

**Remark 2.4.17.** Equivalently, $(\partial\Delta_n)_m = \{\alpha : [m] \to [n] \mid \alpha \text{ is not surjective}\}$.

**Remark 2.4.18.** We have $\partial\Delta_0 = \emptyset$.

**Theorem 2.4.19.** *Take $X$ a simplicial set. Let $N_n$ denote the set of* nondegenerate *$n$-simplices of $X$. For every $n \geq 0$, the diagram*

$$\begin{array}{ccc} \coprod_{\sigma \in N_n} \partial\Delta_n & \longrightarrow & \coprod_{\sigma \in N_n} \Delta_n \\ \downarrow & & \downarrow \\ X(n-1) & \longrightarrow & X(n) \end{array}$$

*is a pushout in* sSet .

*Here, the left vertical arrow is obtained as follows: every $\sigma \in N_n$ gives a map*

$$\sigma : \Delta_n \to X,$$

*which restricts to a map $\partial\sigma : \partial\Delta_n \to X$. As $\partial\Delta_n$ is $(n-1)$-dimensional, we find*

$$\mathrm{Mor}(\partial\Delta_n, X) = \mathrm{Mor}(\partial\Delta_n, X(n-1)),$$

*and hence $\partial\sigma$ can be viewed as a map $\partial\Delta_n \to X(n-1)$.*

**Remark 2.4.20.** As the pushout of an inclusion is an inclusion, we deduce that each map $X(n-1) \hookrightarrow X(n)$ is an inclusion. It follows easily that each $X(n) \hookrightarrow X$ is also an inclusion.

*Proof.* Let $Q$ be the pushout of this diagram. The commutative diagram gives us a natural map $Q \to X(n)$, which we will prove is an isomorphism.

It is easy to see $\dim(Q) \leq n$, as colimits are computed pointwise and $Q$ is the pushout of two simplicial sets of dimension at most $n$. Thus to check that $Q \to X(n)$ is an isomorphism of simplicial sets, we need only show it is a bijection on $k$-simplices for $k \leq n$. But when $k \leq n$, $X(n)_k = X_k$. This makes our life a bit easier.

For $k < n$, we have $Q_k = X(n-1)_k = X_k$, and the resulting map $Q_k \to X(n)_k = X_k$ is just this equivalence $Q_k = X_k$.

For $k = n$, we have

$$Q_n = \left( \coprod_{\sigma \in N_n} (\Delta_n)_n \right) \coprod_A X(n-1)_n,$$

where $A$ is the set of $n$-simplices of $\coprod_{\sigma \in N_n} \partial\Delta_n$.

This coproduct of sets can be simplified: every degenerate $n$-simplex of $\Delta_n$ lies in $\partial\Delta_n$, and hence belongs to $A$. Moreover, no non-degenerate $n$-simplex lies in $A$, and so

$$Q_n = N_n \coprod X(n-1)_n.$$

As $X(n-1)_n$ is exactly the set of degenerate $n$-simplices of $X$ (see Proposition 2.4.12), the map

$$Q_n \to X_n$$

is also a bijection, as desired. □

## 2.5 Braced simplicial sets

**Ideation 2.5.1.** It is very annoying in practice to have to specify where degenerate simplices go in a simplicial set morphism.

We might hope to use Proposition 2.2.9 to get away with avoiding this: define

$$\varphi : X \to Y$$

by saying where the non-degenerate simplices go, and then use Proposition 2.2.9 to determine where the degenerate ones go. That is, we might hope to send every simplicial set to some semisimplicial set of non-degenerate simplices, and then prove that the resulting construction is fully faithful.

Unfortunately, there is an obstruction to doing this: sometimes a face of a non-degenerate simplex is degenerate, and hence the non-degenerate simplices won't form a semisimplicial set. To fix this, we introduce the following notion.

**Definition 2.5.2.** A simplicial set is said to be *braced* if any face of a non-degenerate simplex is non-degenerate.

**Remark 2.5.3.** The standard $n$-simplex $\Delta_n$ is braced, and hence so are any of its simplicial subsets.

**Construction 2.5.4.** Let $X$ be a *braced* simplicial set. We define a semisimplicial set $X^{\mathrm{nd}}$ by putting $X^{\mathrm{nd}}_n$ as the set of *non-degenerate* $n$-simplices of $X$, and using the same face operators as $X$.

**Proposition 2.5.5.** *Let $X$ be a braced simplicial set, and $Y$ an arbitrary simplicial set. Then the restriction map*

$$\mathrm{Mor}_{\mathrm{sSet}}(X, Y) \to \mathrm{Mor}_{\mathrm{semisimplicialSets}}(X^{\mathrm{nd}}, Y)$$

*is a bijection. Here, we interpret $Y$ as a semisimplicial set just by forgetting about the degeneracy morphisms of $Y$ (but keeping the degenerate simplices).*

*Proof.* By Proposition 2.2.9, this restriction map is injective, as every simplex can be written as some combination of degeneracies applied to a non-degenerate simplex.

To see surjectivity, let $\varphi : X^{\mathrm{nd}} \to Y$ be a semisimplicial set morphism. Define

$$\psi : X \to Y$$

as follows: by Proposition 2.2.9, every $\sigma \in X_n$ can be written uniquely as $\sigma = \alpha^*(\tau)$ for $\tau$ non-degenerate; we then define

$$\psi(\sigma) := \alpha^*(\varphi(\tau)).$$

Clearly $\psi$ restricts to $\varphi$ on $X^{\mathrm{nd}}$. To check $\psi$ is a simplicial set map, it suffices to show it is compatible with surjections and injections.

First, fix $\pi : [m] \to [n]$ a monotone surjection; thus $m \geq n$. Take $\sigma \in X_n$. By Proposition 2.2.9, write $\sigma = \alpha^*(\tau)$ with $\tau \in X_k$ non-degenerate and $k \leq n$. As $\pi$ is surjective, clearly $n \leq m$, and so the Proposition 2.2.9 decomposition of $\pi^*\sigma$ is just $(\alpha \circ \pi)^*(\tau)$. Thus

$$\begin{aligned}\psi(\pi^*\sigma) &= (\alpha \circ \pi)^*(\varphi(\tau)) \\ &= \pi^*(\alpha^*\varphi(\tau)) \\ &= \pi^*(\psi(\sigma)),\end{aligned}$$

as desired.

For compatibility with injections, let $i : [m] \hookrightarrow [n]$ be a monotone injection. Take $\sigma = \alpha^*(\tau)$ as before. We can factor $\alpha \circ i : [m] \to [k]$ as

$$\alpha \circ i = j \circ \pi,$$

where $\pi : [m] \to \mathrm{im}(\alpha \circ i)$ is surjective and $j : \mathrm{im}(\alpha \circ i) \hookrightarrow [k]$ is another inclusion. Thus

$$i^*\sigma = \pi^*(j^*\tau).$$

As $j$ is a monotone injection, it can be written as a composition of face maps; as $\tau$ is non-degenerate and $X$ is braced, it follows that $j^*\tau$ is non-degenerate, so $\pi^*(j^*\tau)$ is the Proposition 2.2.9 decomposition of $i^*\sigma$. Using that $\varphi$ is compatible with monotone maps such as $j$, we deduce

$$\begin{aligned}\psi(i^*\sigma) &= \pi^*\varphi(j^*\tau) \\ &= \pi^* j^* \varphi(\tau) \\ &= (j\pi)^*\varphi(\tau) \\ &= (\alpha i)^*\varphi(\tau) \\ &= i^*(\alpha^*\varphi(\tau)) \\ &= i^*\psi(\sigma),\end{aligned}$$

as desired. □

As an example application, we can prove the following result which is a bit annoying to prove directly.

**Corollary 2.5.6.** *Let $X$ be a simplicial set, and $n \geq 2$. Then the function*

$$\alpha : \mathrm{Mor}(\partial\Delta_n, X) \to \prod_{i=0}^{n} \mathrm{Mor}(\Delta_{n-1}, X)$$

*sending $a : \partial\Delta_n \to X$ to its restriction to each non-degenerate $(n-1)$-simplex of $\partial\Delta_n$ is injective. Moreover, its image consists exactly of tuples $(\sigma_0, ..., \sigma_n)$ such that*

$$d^i_{n-1}(\sigma_j) = d^{j-1}_{n-1}(\sigma_i)$$

*for all $i < j$.*

**Remark 2.5.7.** Here, $\sigma_i$ is the face opposite vertex $i$, and the boundary condition is because $\sigma_i \cap \sigma_j$ is the non-degenerate $(n-2)$-simplex not containing $i$ or $j$. Note the $j-1$ needed to make the boundary condition correct!

*Proof.* By Proposition 2.5.5, since $\partial\Delta_n$ is braced, we have

$$\mathrm{Mor}(\partial\Delta_n, X) = \mathrm{Mor}((\partial\Delta_n)^{\mathrm{nd}}, X).$$

As every simplex of the semisimplicial set $(\partial\Delta_n)^{\mathrm{nd}}$ is the face of some $(n-1)$-simplex, our map $\alpha$ is clearly injective.

To get surjectivity onto the claimed image, take $(\sigma_0, ..., \sigma_n)$ obeying our boundary condition.

Define

$$f : (\partial\Delta_n)^{\mathrm{nd}} \to X$$

as follows. A non-degenerate simplex of $\partial\Delta_n$ is just a nonempty *proper* $I \subsetneq [n]$. Let $j \in [n] \setminus I$. Put $\varphi_I$ the inclusion $I \hookrightarrow [n] \setminus \{j\}$. We then define

$$f(I) = \varphi_I^*(\sigma_j).$$

To see that this construction is independent of our choice of $j$, assume $j' \in [n] \setminus I$ is another choice. Let $\psi_I$ be the resulting inclusion; we want to prove

$$\psi_I^*(\sigma_{j'}) = \varphi_I^*(\sigma_j).$$

But this follows immediately from the boundary relationship, as it tells us that $\sigma_{j'}, \sigma_j$ have the same pullback along $[n] \setminus \{j, j'\}$, and hence along $I$.

To see that $f$ is a morphism of semisimplicial sets, let $i : I' \hookrightarrow I$ be an arbitrary monotone inclusion. Then

$$\begin{aligned} f(i^*I) &= f(I') \\ &= \varphi_{I'}^*(\sigma_j) \\ &= i^*\varphi_I^*(\sigma_j), \end{aligned}$$

as long as we pick some $j \notin I$. □

The construction $X \mapsto X^{\mathrm{nd}}$ turns a braced simplicial set into a semisimplicial set. We can similarly turn any semisimplicial set into a braced simplicial set; the construction just interprets the glueing instructions of a semisimplicial set in the category sSet .

**Construction 2.5.8.** Let $S_\bullet$ be a semisimplicial set. We define a braced simplicial set $S_\bullet^+$ as follows. Put

$$S^+(-1) := \emptyset.$$

Inductively define $S^+(n)$ as the pushout

$$\begin{array}{ccc} \coprod_{\sigma \in S_n} \partial\Delta_n & \longrightarrow & \coprod_{\sigma \in S_n} \Delta_n \\ \downarrow & & \downarrow \\ S^+(n-1) & \longrightarrow & S^+(n). \end{array}$$

Here, for $\sigma \in S_n$, the map $\partial\sigma : \partial\Delta_n \to S^+(n-1)$ of simplicial sets is defined as follows. To give a map $\partial\Delta_n \to S^+(n-1)$, it is equivalent to give $n+1$ many $(n-1)$-simplices of $S^+(n-1)$, with some agreement on their faces, by Corollary 2.5.6. And indeed, we just let $\partial\sigma$ be obtained by glueing together the $(n-1)$-simplices $d^i(\sigma) \in S_{n-1} = (S^+(n-1))^{\mathrm{nd}}_{n-1}$.

We claim that $S^+(n)$ is braced. Indeed, let $\sigma \in S^+(n)_i$ be a non-degenerate simplex. We have three cases:

1. if $i < n$, then $\sigma$ is an $i$-simplex of $S^+(n-1)$, and its boundaries agree with its boundaries in $S^+(n-1)$, so all faces of $\sigma$ are non-degenerate as $S^+(n-1)$ is braced;
2. if $i = n$, then $\sigma$ belongs to $S_n$, and its faces are therefore $(n-1)$-simplices of $S^+(n-1)$ coming from $S_{n-1}$, and hence non-degenerate as $S_{n-1} = (S^+(n-1))^{\mathrm{nd}}_{n-1}$;
3. if $i > n$, then there are no non-degenerate $i$-simplices, so there is nothing to check.

Set $S^+ := \mathrm{colim}_n S^+(n)$. This $S^+$ is braced, and has

$$(S^+)^{\mathrm{nd}} \simeq S$$

by construction.

**Proposition 2.5.9.** *The forgetful functor*

$$U : \mathrm{sSet} \to \mathrm{SemisimplicialSets}$$

*has a left adjoint, given by $S \mapsto S^+$.*

*Proof.* The adjunction is furnished by Proposition 2.5.5. □

**Corollary 2.5.10.** *If $X$ is a braced simplicial set, then the inclusion of semisimplicial sets $X^{\mathrm{nd}} \hookrightarrow X$ extends uniquely to an isomorphism $(X^{\mathrm{nd}})^+ \simeq X$.*

*Proof.* By the adjunction, this morphism of semisimplicial sets extends uniquely to a morphism $(X^{\mathrm{nd}})^+ \to X$ of simplicial sets, so the content of this corollary is that this extension is an isomorphism.

For that, it suffices to check this map is a bijection in each degree. It is clearly a bijection on non-degenerate simplices; but both $X$ and $(X^{\mathrm{nd}})^+$ are braced, so every simplex is either non-degenerate or the image under a degeneracy of a lower-dimensional simplex, and so by compatibility of our map with degeneracies we can easily inductively prove that it is a bijection in each degree. □

## 2.6 Connected components

In a point-set topology course, one of the most basic invariants of a topological space is its set of connected components. One can adapt this theory to simplicial sets quite directly.

**Remark 2.6.1.** This section follows Kerodon tag 00G5 quite closely.

**Definition 2.6.2.** A *summand* of a simplicial set $S$ is a simplicial subset $S' \hookrightarrow S$ so that $S = S' \sqcup S''$ for another simplicial subset $S''$ of $S$.

A simplicial set $S$ is *connected* if it has no summands except for $\emptyset$ and $S$.

If $S$ is a simplicial set, then a *connected component* of $S$ is a nonempty connected summand.

**Proposition 2.6.3.** *A sub-simplicial set $S' \subseteq S$ is a summand of $S$ if and only if $(S \setminus S')_n := S_n \setminus S'_n$ defines a simplicial set (meaning that these sets are closed under degeneracies and faces).*

*Proof.* If this is true, then $S = S' \sqcup (S \setminus S')$ and so $S'$ is a summand. If $S'$ is a summand, then $S = S' \sqcup S''$, and so $S''_n = S_n \setminus S'_n$. □

To prove every simplicial set is a disjoint union of its connected components, we first see that there are 'enough' connected components: every simplex lies in one.

**Lemma 2.6.4.** *Let $S$ be a simplicial set, and $x \in S_n$ an $n$-simplex. Let $\mathcal{A}$ be the collection of all summands $T \subseteq S$ so that $x \in T_n$. If we set*

$$S'_i := \bigcap_{T \in \mathcal{A}} T_i,$$

*then $S'$ is a connected simplicial set containing $x$. Furthermore, $S'$ is a connected component of $S$.*

*Here, if $\mathcal{A} = \emptyset$, then $S' := S$.*

*Proof.* Clearly $x \in S'$. To see that $S'$ is a simplicial subset, note that it's closed under face and degeneracy operations: if $y \in S'_i$, then $y \in T_i$ for each $T$, and each $T$ is closed under these operations.

Next we show that $S'$ is a summand. By Proposition 2.6.3, it suffices to show that $S \setminus S'$ is a simplicial subset. And indeed, take $y \in (S \setminus S')_i$. Then $y \notin S'_i$, so there is some $T \in \mathcal{A}$ with $y \notin T_i$. Applying Proposition 2.6.3 to $T$, we deduce that each degeneracy and face of $y$ also lies outside of $T$, and hence inside of $S \setminus S'$. Thus $S \setminus S'$ is stable under degeneracies and boundaries, making it a summand.

Hence $S'$ is a summand. Connectedness is because $S'$ obviously has no non-trivial summands (if it did, it would have a non-trivial summand containing $x$, absurd). □

**Theorem 2.6.5.** *Every simplicial set is the coproduct of all of its connected components, and the connected components are disjoint or equal.*

*Proof.* Let $S$ be a simplicial set. Write $\pi_0(S)$ for its set of connected components.

We first show any two $C, C' \in \pi_0(S)$ are either disjoint or equal. As connected components are summands (by definition), we may write

$$S = C \sqcup (S \setminus C).$$

Then

$$C' = (C \cap C') \sqcup ((S \setminus C) \cap C').$$

But $C'$ is connected, and hence has no nontrivial summands; thus either $C \cap C' = C'$ or $C \cap C' = \emptyset$. When $C \cap C' = C'$, applying this same argument with $C, C'$ swapped also proves $C' \cap C = C$, and hence $C = C'$. Thus $C, C'$ are either disjoint or equal.

It follows that the disjoint union

$$\coprod_{C \in \pi_0(S)} C$$

is a simplicial subset of $S$. To see it is all of $S$, apply Lemma 2.6.4. □

## 2.7 Path components

There is a different notion of connectivity which seems natural to use: we can say that two 0-simplices of $X$ lie in the same component if there is a path of 1-simplices connecting them. This leads to the following definitions.

**Definition 2.7.1.** Let $X_\bullet$ be a simplicial set.

We say that two 0-simplices $x, y \in X_0$ are *naively path equivalent* if there is a 1-simplex $e \in X_1$ such that $d^1(e) = x, d^0(e) = y$.

We say $x, y \in X_0$ are *path equivalent* if they are equivalent under the smallest equivalence relation generated by naive path equivalence.

We let $\pi_0(X_\bullet)$ denote the set of path equivalence classes of $X_0$.

**Proposition 2.7.2.** *Let $S$ be a simplicial set. Then two 0-simplices $x, x' \in S_0$ lie in the same connected component if and only if $[x] = [x']$ in $\pi_0(S_\bullet)$.*

*Proof.* Let $x, x' \in S_0$ be naively path equivalent; take $e \in S_1$ an edge witnessing this. Then $e$ lies in some connected component $C$ of $S$, and so $x, x'$ must lie in $C$ too. As 'lying in the same connected component' is obviously an equivalence relation, it follows that $[x] = [x']$ implies $x, x'$ lie in the same connected component.

For the converse, take $x \in S_0$. Define $T_0$ the set of all vertices in $S_0$ which are equivalent, in $\pi_0(S)$, to $x$. Then let $T_\bullet$ be the simplicial subset of $S$ consisting of simplices whose vertices all lie in $T_0$; thus $T_n \subseteq S_n$ is the set of $n$-simplices all of whose vertices lie in $T_0$. We claim that $T_\bullet$ is a summand of $S_\bullet$.

And indeed, it suffices to show $S_\bullet \setminus T_\bullet$ is a simplicial subset. Take some $\sigma \in S_n \setminus T_n$. Thus $\sigma$ has at least one vertex $x'$ with $[x] \neq [x']$ in $\pi_0(S)$. As $x'$ is a vertex of every degeneracy of $\sigma$, we find that $S_\bullet \setminus T_\bullet$ is then stable under degeneracies.

For any two vertices of a standard $n$-simplex like $\sigma$, there is some edge connecting them; hence all vertices of $\sigma$ are equal in $\pi_0(S)$, and so *all* vertices of $\sigma$ are unequal to $[x]$. Thus every face of $\sigma$ also contains a vertex lying outside of $T_0$, and so $S_\bullet \setminus T_\bullet$ is stable under faces, making it a simplicial subset.

This summand $T_\bullet$ is also connected. Indeed, assume

$$T_\bullet = T'_\bullet \sqcup T''_\bullet.$$

Say $x \in T'_0$. For every $x' \in T''_0$, there is some chain of edges in $T_1$ connecting $x$ to $x'$; label these edges $e_0, e_1, ..., e_\ell$, so that $x'$ is a vertex of $e_\ell$ and $x$ is a vertex of $e_0$. Each $e_i$ belongs to exactly one of $T'_1$ or $T''_1$. But as $T', T''$ are both stable under taking faces, we find $e_\ell \in T''_1$ but $e_0 \in T'_1$. But now we have a contradiction: $e_0, e_1$ share a vertex, and hence $e_1 \in T'_1$ as well; continuing, we deduce $e_\ell \in T'_1$, absurd since $T''_1 \cap T'_1 = \emptyset$.

Thus $T''_\bullet = \emptyset$, and so $T_\bullet$ is connected; this finishes the claim. □

**Remark 2.7.3.** Not all connected point-set topological spaces are path connected. However, all connected simplicial sets are path connected, as we just saw! Thus simplicial sets allow us to avoid a pathology of point-set topology; this pathology is often of use to analysts, but for geometers and algebraic topologists, eliminating this pathology makes life easier without losing any important examples.

As a corollary, with this we can easily prove that the image of a connected simplicial set is connected.

**Construction 2.7.4** (Image of a simplicial set morphism)**.** Let $f : S \to T$ be a morphism of simplicial sets. We write $f(S)$ for the simplicial subset of $T$ given by

$$f(S)_n = \{f(\sigma) \mid \sigma \in S_n\}.$$

The compatibility of $f$ with faces and degeneracies implies $f(S)$ is actually a simplicial subset.

**Corollary 2.7.5.** *If $f : S \to T$ is a morphism of simplicial sets, and $S$ is connected, then $f(S)$ is connected as well.*

*Proof.* Choose $f(s), f(s') \in f(S)_0$. Then there is some chain of edges $e_0, ..., e_n \in S_1$ connecting $s$ to $s'$; the 1-simplices $f(e_0), f(e_1), ..., f(e_n)$ witness $[f(s)] = [f(s')]$ in $\pi_0(f(S))$, so that $f(S)$ is connected. □

Relatedly, we have the following useful construction.

**Construction 2.7.6** (Functoriality of $\pi_0$)**.** Let $f : X \to Y$ be a morphism of simplicial sets. We define

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y)$$

by

$$\pi_0(f)([x]) := [f(x)].$$

To see this is well-defined, it suffices to check that if $x, x'$ are naively path equivalent, then $f(x), f(x')$ are as well; but this is obvious: if $e \in X_1$ is an edge from $x$ to $x'$, then $f(e)$ is an edge from $f(x)$ to $f(x')$.

**Remark 2.7.7.** Using Corollary 2.7.5, we can describe $\pi_0(f)$ another way. Write

$$X = \coprod_{i \in \pi_0(X)} X_i,$$

for $X_i$ the connected components of $X$. Then each $f(X_i)$ is a connected subset of $Y$ by Corollary 2.7.5, and hence is contained in a unique connected component $Y_j$ of $Y$; we have

$$\pi_0(f)(i) := j.$$

# Chapter 3

# Geometric realization

## 3.1 Introduction

Simplicial sets are instructions on how to glue together simplices to get a shape. In this chapter, we construct a functor that implements those glueing instructions in the category of topological spaces.

That is, we desire a functor

$$\| - \| : \mathrm{sSet} \to \mathrm{Top}$$

sending a simplicial set to the 'same' shape, but modelled as a point-set topological space.

We know that $\|\Delta_n\|$ ought to be the standard topological $n$-simplex. We also know that every simplicial set should be obtained by glueing various simplices together, and that geometric realization should respect glueing. Every simplicial set can be obtained as a colimit of standard simplices (using the skeletal filtration), so knowing how our functor behaves on standard simplices ought to uniquely determine it. To formalize this idea, we recall in Section 3.2 some general categorical machinery.

## 3.2 Presheaf categories as free cocompletions

Simplicial sets are defined as the category of functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$ . Categories $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ are called *presheaf* categories, and they always have the amusing property that every object in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is a colimit of objects in the image of the Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

**Lemma 3.2.1.** *Let $\mathcal{C}$ be an essentially small category, and $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a presheaf. Define a diagram category $I$ whose objects are pairs $(X \in \mathcal{C}, x \in \mathcal{F}(X))$, with a map $(X, x) \to (Y, y)$ being a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ such that $\mathcal{F}(f)(y) = x$.*

*Then the map*

$$\mathrm{colim}_{(X,x)\in I}\, h_X \to \mathcal{F},$$

*sending the $(X, x)$ copy of $h_X$ to the map $h_X \xrightarrow{x} \mathcal{F}$, is an equivalence.*

*Proof.* We can explicitly compute that

$$\left(\mathrm{colim}_{(X,x)\in I}\, h_X\right)(Y) = \mathrm{colim}_{(X,x)\in I}\, \mathrm{Mor}_{\mathcal{C}}(Y, X).$$

We want to check that the map

$$\operatorname{colim}_{(X,x)\in I} \operatorname{Mor}_{\mathcal{C}}(Y,X) \to \mathcal{F}(Y)$$

is a bijection of sets, where on the $(X,x)$ factor, the map $\operatorname{Mor}(Y,X) \to \mathcal{F}(Y)$ sends $f: Y \to X$ to $\mathcal{F}(f)(x)$.

This map is obviously surjective, by considering the $\operatorname{Mor}(Y,Y)$ factors indexed by $(Y,y)$ and varying $y$. To see injectivity, suppose that $f_{X,x} \in \operatorname{Mor}(Y,X), g_{X',x'} \in \operatorname{Mor}(Y,X')$ map to the same point $y$ of $\mathcal{F}(Y)$. Then $\mathcal{F}(f_{X,x})(x) = \mathcal{F}(g_{X',x'})(x') = y$. Then there are maps $(Y,y) \to (X,x), (Y,y) \to (X',x')$ in $I$ such that $f_{X,x}, g_{X',x'}$ become identified inside $(Y,y)$, proving the injectivity. □

As Lemma 3.2.1 *canonically* writes every object of $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ as a colimit of objects of $\mathcal{C}$, we can use it to prove every functor $\mathcal{C} \to \mathcal{D}$ extends to a unique colimit preserving functor $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{D}$.

**Theorem 3.2.2.** *Let $\mathcal{C}$ be a small category. Then the category $\mathcal{P} := \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is the cocompletion of $\mathcal{C}$. That is, $\mathcal{P}$ is cocomplete, and obeys the universal property that: for any cocomplete category $\mathcal{D}$, if $\operatorname{Fun}'(\mathcal{P},\mathcal{D})$ is the category of functors which commute with all colimits, then the map*

$$\operatorname{Fun}'(\mathcal{P},\mathcal{D}) \xrightarrow{J^*} \operatorname{Fun}(\mathcal{C},\mathcal{D})$$

*is an equivalence of categories, where $J: \mathcal{C} \to \mathcal{P}$ is the Yoneda embedding.*

*Proof.* It's clear that $\mathcal{P}$ is cocomplete, as we can take colimits pointwise and Set is cocomplete.

By Lemma 3.2.1, every object of $\mathcal{P}$ is canonically a colimit of objects of $\mathcal{C}$. Thus, any natural transformation $\eta: F \to G$ between cocontinuous functors $F, G: \mathcal{P} \to \mathcal{D}$ is determined by what it does on the objects $h_X$, as for any $\mathcal{F} \in \mathcal{P}$, the commutativity of the squares

$$\begin{array}{ccc} F(h_X) & \xrightarrow{\eta_{h_X}} & G(h_X) \\ \downarrow{\scriptstyle Fx} & & \downarrow{\scriptstyle Gx} \\ F(\mathcal{F}) & \xrightarrow{\eta_{\mathcal{F}}} & G(\mathcal{F}) \end{array}$$

ranging over all $X \in \mathcal{C}, x \in \mathcal{F}(X)$ implies that

$$\eta_{\mathcal{F}} : \operatorname{colim}_{X\in\mathcal{C}, x\in\mathcal{F}(X)} F(h_X) \to G(\mathcal{F})$$

is determined by the maps $\eta_{h_X}$. Thus $J^*$ is faithful. Fullness is similar: any map $\eta: J^*F \to J^*G$ induces a natural transformation $\eta: F \to G$ by the same colimit formula.

For essential surjectivity, given any functor $F: \mathcal{C} \to \mathcal{D}$, we define $F': \mathcal{P} \to \mathcal{D}$ on objects by the formula

$$F'(\mathcal{F}) := \operatorname{colim}_{(X\in\mathcal{C}, x\in\mathcal{F}(X))} F(X),$$

and we send a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ to the obvious map

$$\operatorname{colim}_{(X,x\in\mathcal{F}(X))} F(X) \to \operatorname{colim}_{(X,x\in\mathcal{G}(X))} F(X)$$

which, on the $(X, x \in \mathcal{F}(X))$ factor, is the inclusion

$$F(X) \xrightarrow{(X,\varphi(X)(x))} \operatorname{colim}_{(X,x\in\mathcal{G}(X))} F(X).$$

As colimits commute with colimits, this functor $F'$ commutes with colimits, and hence belongs to $\mathrm{Fun}'(\mathcal{P}, \mathcal{D})$. As $\mathrm{id}_X : X \to X$ is a final object in the diagram category of the presheaf $h_X$, we deduce that

$$F'(h_X) = \mathrm{colim}_{X' \to X} F(X') \simeq F(X),$$

naturally in $X$, and so $J^*(F') \simeq F$, as desired. □

Specializing to sSet, we deduce the following.

**Corollary 3.2.3.** *Let $\mathcal{C}$ be a cocomplete category. Then colimit preserving functors* $\mathrm{sSet} \to \mathcal{C}$ *are equivalent to functors $\Delta \to \mathcal{C}$.*

**Definition 3.2.4.** A *cosimplicial object* of a category $\mathcal{C}$ is a functor $Q^\bullet : \Delta \to \mathcal{C}$.

**Remark 3.2.5.** By Corollary 3.2.3, colimit preserving functors $\mathrm{sSet} \to \mathcal{C}$ are equivalent to cosimplicial objects. As cosimplicial objects are quite easy to specify, we will typically define functors out of sSet using them.

## 3.3 Constructing geometric realization

**Definition 3.3.1.** Let $\mathcal{C}$ be a cocomplete category. If $Q^\bullet : \Delta \to \mathcal{C}$ is a cosimplicial object of $\mathcal{C}$, we denote by

$$\| - \|_{Q^\bullet} : \mathrm{sSet} \to \mathcal{C}$$

the functor associated to $Q^\bullet$ by Corollary 3.2.3. We call this the *geometric realization* along $Q^\bullet$.

**Construction 3.3.2** (The standard cosimplicial object)**.** Let $\Delta^n$ denote the *topological* $n$-simplex, defined as the closed subset of $[0,1]^{n+1}$ cut out by the equation $x_0 + \cdots + x_n = 1$.

We define the *standard cosimplicial object* to be the cosimplicial topological space

$$\Delta^\bullet : \Delta \to \mathrm{Top}$$

which sends $[n]$ to $\Delta^n$, and sends an order preserving map $\varphi : [n] \to [m]$ to the continuous function

$$\Delta^\bullet(\varphi)(x_0, ..., x_n) = \left( \sum_{i \in \varphi^{-1}(0)} x_i, \sum_{i \in \varphi^{-1}(1)} x_i, ..., \sum_{i \in \varphi^{-1}(m)} x_i \right).$$

**Notation 3.3.3.** We write $\| - \| : \mathrm{sSet} \to \mathrm{Top}$ for the geometric realization with respect to the standard cosimplicial object.

**Remark 3.3.4.** As geometric realization commutes with arbitrary colimits, the skeletal filtration (Section 2.4.2) allows us to get a rather concrete description of the topological space associated to a simplicial set, as a colimit of topological spaces $T_n$ where $T_{n+1}$ is obtained from $T_n$ by attaching various $(n+1)$-simplices to $T_n$.

## 3.4 The simplicial set associated to a topological space

As a general principle, one should expect any functor that preserves colimits to be a left adjoint; this isn't always true – the adjoint functor theorem has some hypotheses – but it's a very good rule of thumb. In particular, we should investigate whether or not geometric realization is a left adjoint.

**Ideation 3.4.1.** Let $Q^\bullet$ be some cosimplicial object of a category $\mathcal{C}$, and $X$ a simplicial set, and $Y \in \mathcal{C}$. The right adjoint to geometric realization along $Q^\bullet$ should send $Y$ to some simplicial set $Y'$ having the property that maps $X \to Y'$ are the same as maps $\|X\|_{Q^\bullet} \to Y$.

Simplicial sets are determined by how simplices map into them, though; this suggests defining a simplicial set $Y'$ by

$$Y'_n := \mathrm{Mor}(Q^n, Y),$$

leading to the following.

**Definition 3.4.2.** Let $\mathcal{C}$ be a category, and $Q : \Delta \to \mathcal{C}$ a *co*simplicial object. We define

$$\mathrm{Sing}^Q : \mathcal{C} \to \mathrm{sSet}$$

as the functor

$$\mathrm{Sing}^Q(X)_n := \mathrm{Mor}_{\mathcal{C}}(Q^n, X),$$

using the coboundary and codegeneracy maps on $Q$ to create the boundary and degeneracy maps on $\mathrm{Sing}^Q$.

**Proposition 3.4.3.** *Let $Q : \Delta \to \mathcal{C}$ be a cosimplicial object in a cocomplete category. Then $\| - \|_Q, \mathrm{Sing}^Q$ are an adjoint pair of functors.*

*Proof.* Let $S \in \mathrm{sSet}, X \in \mathcal{C}$. Then

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(\|S\|_Q, X) &= \mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}_{(n,\sigma\in S_n)}\, Q^n, X) \\
&= \lim_{(n,\sigma\in S_n)} \mathrm{Mor}_{\mathcal{C}}(Q^n, X) \\
&= \lim_{(n,\sigma\in S_n)} \mathrm{Mor}_{\mathrm{sSet}}(\Delta_n, \mathrm{Sing}^Q(X)) \\
&= \mathrm{Mor}_{\mathrm{sSet}}(\mathrm{colim}_{(n,\sigma\in S_n)}\, \Delta_n, \mathrm{Sing}^Q(X)) \\
&= \mathrm{Mor}_{\mathrm{sSet}}(S, \mathrm{Sing}^Q(X)).
\end{aligned}$$

The constructed bijection is functorial in $(S, X)$, and thus gives us our desired adjunction. □

**Remark 3.4.4.** We write

$$\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$$

for the right adjoint to the standard geometric realization functor. This functor associates to any topological space $X$ a simplicial set where

1. *every* point of $X$ is a vertex,
2. *every* continuous path in $X$ is a 1-simplex,
3. *every* continuous map $\Delta^2 \to X$ is a 2-simplex,

4. etc.

This is the *maximal possible triangulation* of $X$: everything is a triangle! Of course, this is a very poorly behaved triangulation, which is why it is often called a *singular* triangulation: the degenerate simplices of $\mathrm{Sing}(X)$ correspond to degenerate 'triangles.'

## 3.5 The connected components of a geometric realization

Geometric realization preserves connected components.

**Definition 3.5.1.** Let $X$ be a point-set topological space. A *connected component* of $X$ is a nonempty connected subset of $X$ which is not contained in any other connected subset; connected components end up being disjoint or equal, and we write $\pi_0(X)$ for the set of connected components of $X$.

**Theorem 3.5.2.** *Let $S$ be a simplicial set. Then, the following are equivalent:*

1. *$|S|$ is path-connected,*
2. *$|S|$ is connected,*
3. *$S$ is connected.*

*Proof.* When $|S|$ is path-connected, it is of course connected.

Assume now that $|S|$ is connected. Then $S$ must be connected: otherwise, we could write $S = T_1 \sqcup T_2$ as a disjoint union of two non-empty simplicial sets, and hence

$$|S| = |T_1| \coprod |T_2|$$

is disconnected as a point-set topological space.

Finally, assume $S$ is connected. For each $n \geq 1$, the simplicial subset $S(n) = \mathrm{sk}_n(S)$ is therefore connected as well, as connectivity depends only on 1-simplices by Proposition 2.7.2.

We first claim that $|S(1)|$ is path-connected. Letting $N$ denote the set of non-degenerate 1-simplices of $S(1)$, we have a pushout diagram

$$\begin{array}{ccc} \coprod_{e \in N} \{0,1\} & \longrightarrow & \coprod_{e \in N} \Delta_1 \\ \downarrow & & \downarrow \\ S(0) & \longrightarrow & S(1). \end{array}$$

In particular, we have a pushout diagram of point-set topological spaces

$$\begin{array}{ccc} \coprod_{e \in N} \{0,1\} & \longrightarrow & \coprod_{e \in N} [0,1] \\ \downarrow & & \downarrow \\ |S(0)| & \longrightarrow & |S(1)|. \end{array}$$

Let $y \in |S(1)|$. If $y$ lies in the image of

$$\coprod_{e \in N} [0,1] \to |S(1)|,$$

then say $y$ is the image of $(e,t)$. The continuous path from $t$ to 0 in $[0,1]$ then gives a path in $|S(1)|$ from $y$ to a point in $|S(0)|$, so every point lies in the path component of some point of $|S(0)|$.

To see that all points of $|S(0)|$ lie in the same path component, take two points $x, y \in |S(0)|$. By Proposition 2.7.2, there is then some sequence of edges $e_0, e_1, ..., e_\ell \in S_1$ such that $e_0$ has $x$ as a vertex, $e_\ell$ has $y$ as a vertex, and $e_i, e_{i+1}$ share a vertex in common. Each edge $e_i$ gives a continuous path between its vertices in $|S(1)|$; concatenating these paths, we obtain a path from $x$ to $y$, as desired.

We now prove that, if $|S(n)|$ is path-connected, then $|S(n+1)|$ is path-connected as well. For this, note that we can build the geometric realization of $|S(n+1)|$ as the pushout

$$\begin{array}{ccc} \coprod_{\sigma \in N_{n+1}} \partial \Delta^{n+1} & \longrightarrow & \coprod_{\sigma \in N_{n+1}} \Delta^{n+1} \\ \downarrow & & \downarrow \\ |S(n)| & \longrightarrow & |S(n+1)|. \end{array}$$

Take $x, y \in |S(n+1)|$.

If $x, y$ both lie in the same $(n+1)$-simplex $\Delta^{n+1}$, or both lie in $|S(n)|$, then there is clearly a path between them. If $x \in |S(n)|$ and $y$ lies in some $(n+1)$-simplex, then form a path from $y$ to a point on the boundary of that simplex; that boundary lies in $|S(n)|$, so $x, y$ lie in the same path component. Similarly, if $x, y$ lie in two different simplices, then draw paths to their boundaries and then draw a path (in $|S(n)|$) from those two boundary points.

Thus $|S| = \operatorname{colim}_n |S(n)|$ is a filtered colimit of path-connected spaces; thus it is path-connected: if $x, y \in |S|$, then we can find some $n \geq 1$ with $x, y \in |S(n)|$, and then draw a path between them in $|S(n)|$ to get a path between them in $|S|$. □

**Construction 3.5.3.** Let $S$ be a simplicial set. We construct (functorially in $S$) a bijection

$$\alpha_S : \pi_0(S) \to \pi_0(|S|).$$

Writing

$$S = \coprod_{i \in \pi_0(S)} S_i$$

as a disjoint union of connected components, Theorem 3.5.2 implies each $|S_i|$ is connected, so we put $\alpha_S(i) = |S_i|$. Functoriality in $S$ follows easily from Remark 2.7.7.

The compatibility of geometric realization with coproducts implies

$$|S| = \coprod_{i \in \pi_0(S)} |S_i|,$$

so that $\alpha_S$ is a bijection.

There is a related result about the counit.

**Theorem 3.5.4.** *Let $X$ be a topological space.*

*Then the continuous map*

$$\epsilon : |\operatorname{Sing}(X)| \to X$$

*induces a bijection*

$$\pi_0(|\operatorname{Sing}(X)|) \simeq \textit{path components of } X.$$

*Proof.* To begin, we first prove that every $x \in |\operatorname{Sing}(X)|$ lies in the same path component as a point in the image of the (possibly discontinuous!) map

$$j : X = |\operatorname{sk}_0(\operatorname{Sing}(X))| \to |\operatorname{Sing}(X)|.$$

Here, the equality $X = |\operatorname{sk}_0(\operatorname{Sing}(X))|$ is only a bijection of sets, and is only a homeomorphism when $X$ is discrete.

Anyways, to prove this, note that

$$|\operatorname{Sing}(X)| = \operatorname{colim}_{(n,\sigma \in \operatorname{Sing}(X)_n)} \Delta^n.$$

Suppose $x \in |\operatorname{Sing}(X)|$ lies in the image of the continuous map

$$i_\sigma : \Delta^n \to |\operatorname{Sing}(X)|$$

associated to $\sigma \in \operatorname{Sing}(X)_n$. Write $x = i_\sigma(x')$. As $\Delta^n$ is path connected, $x$ then lies in the same path component as any point in the image of $i_\sigma$. In particular, $x$ lies in the same path component as $i_\sigma(0)$; letting $\varphi : [0] \hookrightarrow [n]$ be the inclusion onto 0, we find

$$i_\sigma(0) = i_{\varphi^* \sigma}(0) = j(\sigma(0)),$$

where in the last step we view $\sigma$ as a continuous map $\Delta^n \to X$. Thus $x$ is in the same path component as a point of the image of $j$.

Now we analyze the desired bijectivity on $\pi_0$. Surjectivity is clear, as $\epsilon$ itself is surjective.

For injectivity, suppose $x, x' \in |\operatorname{Sing}(X)|$ are such that $\epsilon(x), \epsilon(x')$ lie in the same path component of $X$. Thus there is a path

$$p : [0,1] \to X$$

from $\epsilon(x)$ to $\epsilon(x')$.

By our preamble, we may assume $x, x'$ lie in the image of $j$. The path $p$ then defines a 1-simplex of $\operatorname{Sing}(X)$ having endpoints $j(\epsilon(x)) = x$ and $j(\epsilon(x')) = x'$. Therefore $x, x'$ lie in the same path component of $|\operatorname{Sing}(X)|$, as desired. □

## 3.6 Geometric realizations and products

Geometric realization is a left adjoint, and hence commutes with all colimits. But by a miracle, geometric realization *also* commutes with all finite limits, at least at the level of underlying sets.

### 3.6.1 Drinfeld's construction of the geometric realization

As you might suspect, Drinfeld proves his claim by showing that the geometric realization can be given by a filtered colimit (at the moment it is just a colimit). His approach is based on a different way to realize the standard $n$-simplex. While using standard $n$-simplices is more natural from the point of view of the geometric motivation of simplicial sets, if one starts with the abstract category $\Delta^{\mathrm{op}}$, then it is simpler to think of the following cosimplicial space.

**Construction 3.6.1.** Let

$$\mathrm{Conf}^n := \{(x_1, ..., x_n) \in [0,1]^n \mid x_1 \leq \cdots \leq x_n\}$$

be the set of *monotone* sequences in the cube, viewed as a closed subspace of the standard $n$-cube. We can assemble these spaces into a cosimplicial space $\mathrm{Conf}^\bullet$, with a monotone map $\varphi : [n] \to [m]$ being sent to the continuous function

$$(x_1, ..., x_n) \mapsto (y_1, ..., y_m),$$

$$y_i = x_{\#\{j \mid \varphi(j) < i\}}.$$

By convention, in the above formula we interpret $x_0$ as 0 and $x_{n+1}$ as 1.

**Remark 3.6.2.** This space $\mathrm{Conf}^n$ is also the configuration space of $n$ (unlabelled) points in $[0,1]$.

**Ideation 3.6.3.** A point of $\Delta^n$ is a tuple of non-negative numbers $(t_0, ..., t_n)$ which add to 1. A point of $\mathrm{Conf}^n$ is a monotone sequence $x_1 \leq \cdots \leq x_n$. What is the relation between these two points?

Visualize a point of $\mathrm{Conf}^n$ as $n$ points in the unit interval. Notice how they divide $[0,1]$ up into $n+1$ subintervals; we can record these intervals either by

1. writing down the $n$ points, or
2. writing down the *lengths* of each subinterval.

Thus $\mathrm{Conf}^n \simeq \Delta^n$, as they are both the moduli space of subdivisions of $[0,1]$ into $n+1$ subintervals. Writing this out concretely, we get the following.

**Lemma 3.6.4.** *The continuous maps*

$$\alpha_n : \Delta^n \to \mathrm{Conf}^n,$$

$$\alpha_n(t_0, ..., t_n) = (t_0, t_0 + t_1, ..., t_0 + \cdots + t_{n-1})$$

*are all homeomorphisms, and assemble into an isomorphism of cosimplicial spaces* $\mathrm{Conf}^\bullet \simeq \Delta^\bullet$.

*Proof.* The inverse to $\alpha_n$ is

$$\beta_n(x_1, ..., x_n) = (x_1, x_2 - x_1, ..., x_n - x_{n-1}, 1 - x_n).$$

Thus $\alpha_n$ is a homeomorphism.

Take $\varphi : [n] \to [m]$ a monotone map. We wish to check that the diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\alpha_n} & \mathrm{Conf}^n \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \Delta^m & \xrightarrow{\alpha_m} & \mathrm{Conf}^m \end{array}$$

commutes. This follows from a direct computation which we leave to the reader. □

**Remark 3.6.5.** Currently, the cosimplicial structure on $\mathrm{Conf}^\bullet$ is a little awkward, primarily because

$$\mathrm{Conf}^n = \mathrm{Mor}([n-1], I)$$

(at least at the level of underlying sets, where here $I = [0,1]$ and we're looking at monotone maps), and so it is somewhat awkward to turn a map $\varphi : [n] \to [m]$ into a map $\mathrm{Conf}^n \to \mathrm{Conf}^m$ (as $\mathrm{Mor}(-, I)$ is contravariant).

Drinfeld observed that these maps $\mathrm{Conf}^n \to \mathrm{Conf}^m$ become more natural if one writes

$$\mathrm{Conf}^n = \mathrm{Mor}'(I, [n]),$$

where now we look at monotone maps $[0,1] \to \{0, 1, ..., n\}$, modulo the relation that two maps are equal if they differ at finitely many points; this is because a monotone map $[0,1] \to \{0, 1, ..., n\}$ is the same as a partition of $[0,1]$ into $n+1$ intervals, and up to the information of which intervals include which endpoints, this is the same data as a monotone sequence of $n$ elements in $[0,1]$.

Equivalently, one can replace $[0,1]$ with

$$\tilde{I} := \text{“}\varprojlim_{F \subseteq I}\text{”} \pi_0(I \setminus F),$$

viewed as a pro-object in $\Delta$, where the inverse limit is taken over all finite subsets $F$ of $I$. Then maps

$$\tilde{I} \to [n]$$

in $\mathrm{Pro}(\Delta)$ are the same as points of $\mathrm{Conf}^n$, because

$$\mathrm{Mor}(\tilde{I}, [n]) = \mathrm{colim}_{F \subseteq I} \mathrm{Mor}(\pi_0(I \setminus F), [n]).$$

Then $\mathrm{Conf}^\bullet$, at least at the level of underlying sets, is the cosimplicial set $\mathrm{Mor}(\tilde{I}, -)$.

Remark 3.6.5 also gives us a simpler formula for the underlying set of geometric realization; recall that cocontinuous functors out of sSet are the same as cosimplicial objects, and so we should expect an equivalence of cosimplicial objects to give an interesting result about functors.

**Lemma 3.6.6.** *The functors* $\mathrm{sSet} \to \mathrm{Set}$ *given by*

$$X \mapsto |X|,$$

$$X \mapsto \mathrm{colim}_{F \subseteq I} X(\pi_0(I \setminus F))$$

*are naturally isomorphic.*

*Proof.* These two functors are both cocontinuous, so it suffices to prove that the associated cosimplicial sets are isomorphic. But this was the content of Remark 3.6.5. □

### 3.6.2 Applications to finite limits

**Lemma 3.6.7.** *The functor* sSet $\to$ Set *sending a simplicial set to the* underlying set *of its geometric realization preserves all finite limits.*

*Proof.* By Lemma 3.6.6, it is enough to prove that the diagram of finite subsets of $[0,1]$ is filtered; and indeed, the union of two finite sets is still finite. □

Lemma 3.6.7 is intriguing, but not as useful as we'd desire, since the geometric realization is naturally a topological space and not a set. Fortunately, we can upgrade Lemma 3.6.7 to a statement in Top, as long as we put some mild assumptions on our simplicial sets. The core of the argument is to use that a map of compact Hausdorff spaces is a homeomorphism if and only if the associated function on sets is a bijection. This lets us check that finite limits of *finite* simplicial sets will commute with geometric realization, even in Top . The most useful special case of this paradigm is the following.

**Definition 3.6.8.** A *finite* simplicial set is a simplicial set with only finitely many non-degenerate simplices.

**Theorem 3.6.9.** *Let $X$ be a finite simplicial set and $Y$ a simplicial set. Then the natural map*

$$|X \times Y| \to |X| \times |Y|$$

*is a homeomorphism of topological spaces.*

*Proof.* When $X, Y$ are both finite, then $|X \times Y|$ and $|X| \times |Y|$ are both compact Hausdorff spaces, so this follows by Lemma 3.6.7.

In general, we may write $Y = \operatorname{colim}_{Y'} Y'$ as the filtered colimit of its *finite* simplicial subsets $Y'$ (as a union of finite subsets is still finite, this limit will be filtered). Then

$$|X \times Y| = \operatorname{colim}_{Y'} |X \times Y'| \simeq \operatorname{colim}_{Y'} |X| \times |Y'| = |X| \times |Y|,$$

as desired. □

## 3.7 The nerve of a category

We end this chapter by giving another example of an interesting cosimplicial object; in fact, we give here a cosimplicial *category*.

The basic simplices are the totally ordered sets $[n]$. But totally ordered sets can also canonically be viewed as categories; in fact, any poset gives rise to a category (the objects are the elements of the poset, and we have an arrow from $x$ to $y$ whenever $x \leq y$).

Unwinding the construction of singular functor associated to this cosimplicial category, we get the following functor Cat $\to$ sSet .

**Construction 3.7.1** (Nerve of a category)**.** Let $\mathcal{C}$ be a small category. Its *nerve* is the simplicial set $N(\mathcal{C})$ defined such that

1. $N(\mathcal{C})_n$ is the set of composable morphisms

$$x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} x_n$$

in $\mathcal{C}$ (so, an element of $N(\mathcal{C})_n$ is a tuple $(x_0, ..., x_n, a_0, ..., a_{n-1})$, where the $x_i$ are objects of $\mathcal{C}$ and $a_i$ are morphisms $x_i \to x_{i+1}$); equivalently, if we write $[n]$ for the poset category associated to $\{0, 1, ..., n\}$ (so $\mathrm{Mor}(i, j)$ is either empty if $i > j$, or contains a unique element if $i \leq j$), then $N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$;

2. for an order-preserving map $\varphi : [n] \to [m]$, we view $\varphi$ as a functor of the poset categories, and let $N(\mathcal{C})(\varphi) : N(\mathcal{C})_m \to N(\mathcal{C})_n$ be precomposition with $\varphi$.

**Proposition 3.7.2.** *The nerve functor*

$$N : \mathrm{Cat} \to \mathrm{sSet}$$

*is fully faithful.*

*Proof.* Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. We wish to prove that the natural map

$$\mathrm{Fun}(\mathcal{C}_1, \mathcal{C}_2) \to \mathrm{Mor}_{\mathrm{sSet}}(N(\mathcal{C}_1), N(\mathcal{C}_2))$$

is a bijection. The injectivity is obvious, as we can recover what a functor $F$ does on objects and arrows from $N(F)$, and $F$ is uniquely determined by such data.

For surjectivity, take $\varphi : N(\mathcal{C}_1) \to N(\mathcal{C}_2)$ a map of simplicial sets. Define a functor

$$F : \mathcal{C}_1 \to \mathcal{C}_2$$

on objects as

$$F(x) = \varphi_0(x)$$

and on arrows as

$$F(a) = \varphi_1(a),$$

using that $N(\mathcal{C}_1)_0$ is the set of objects of $\mathcal{C}_1$ and $N(\mathcal{C}_1)_1$ is its set of arrows. The compatibility of $\varphi$ with boundary operators shows that $F(a)$ has the correct source and target.

The compatibility of $\varphi$ with degeneracies shows $F$ sends identity arrows to identity arrows. The functoriality of $F$ follows from the fact that compositions are witnessed by 2-simplices, and $\varphi$ is compatible with boundaries and sends 2-simplices to 2-simplices. To see $N(F) = \varphi$, just observe that the simplices in the nerve of a category are determined by their boundary edges, so the agreement of $N(F)$ with $\varphi$ on 0- and 1-simplices implies the agreement on all higher simplices. □

**Lemma 3.7.3.** *Consider the cosimplicial category* $[\bullet] : \Delta \to \mathrm{Cat}$ *where*

1. $[\bullet]([n]) = [n]$, *the category associated to the poset* $0 < 1 < \cdots < n$,
2. $[\bullet](f)$ *is the obvious functor associated to the monotone function* $f$.

*Then there is a natural isomorphism (of functors* $\mathrm{Cat} \to \mathrm{sSet}$*)*

$$N \simeq \mathrm{Sing}^{[\bullet]}.$$

*In particular, $N$ is a right adjoint.*

*Proof.* By definition,

$$N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}),$$
$$\mathrm{Sing}^{[\bullet]}(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}),$$

and so we conclude. □

**Notation 3.7.4.** We write $\mathfrak{C}[S] := \|S\|_{[\bullet]}$, and refer to $\mathfrak{C}[S]$ as the *categorical realization* of $S$.

**Ideation 3.7.5.** Intuitively, the nerve $N(\mathcal{C})$ of a category $\mathcal{C}$ is determined by its 2-skeleton: indeed, the 0-simplices encode objects, the 1-simplices encode morphisms, the 2-simplices encode composition, and a category contains no more data.

One might suspect $N(\mathcal{C}) = i_{2!}\,\mathrm{sk}_2(N(\mathcal{C}))$, but this is *false* in general; the nerve can have non-degenerate 3-simplices, but $i_{2!}$ will never produce a simplicial set with non-degenerate 3-simplices.

One way to guess that this would be wrong is to observe that $i_{2!}$, being a left adjoint, is defined by a certain colimit; whereas $N(\mathcal{C})$ is defined by a certain limit.

But recall that $\mathrm{sk}_2$ *also* had a right adjoint, $\mathrm{cosk}_2$. Fortunately, it is true that $N(\mathcal{C}) = \mathrm{cosk}_2(\mathrm{sk}_2(N(\mathcal{C})))$.

**Proposition 3.7.6.** *Let $\mathcal{C}$ be a category. Then its nerve $N(\mathcal{C})$ is 2-coskeletal; that is, the unit map*

$$N(\mathcal{C}) \to \mathrm{cosk}_2\,\mathrm{sk}_2\,N(\mathcal{C})$$

*is an isomorphism of simplicial sets.*

*Proof.* Set $X = N(\mathcal{C})$. By definition,

$$\mathrm{cosk}_2(X)_n := \lim_{[j]\to[n], j\leq 2} X_j.$$

For injectivity of

$$N(\mathcal{C})_n \to \lim_{[j]\to[n], j\leq 2} X_j,$$

observe that every $n$-simplex of $N(\mathcal{C})$ is determined by its 1-skeleton.

For surjectivity, suppose we have $\sigma \in \mathrm{cosk}_2(X)_n$. Let $e_i : [1] \to [n]$ be the map $e_i(0) = i, e_i(1) = i+1$. Then $e_i^*(\sigma) \in X_1$ is some morphism in $\mathcal{C}$. Similarly, for each $0 \leq i \leq n$, we get objects $x_i := i^*\sigma$ of $\mathcal{C}$. These define for us an $n$-simplex

$$x_0 \xrightarrow{e_0^*(\sigma)} x_1 \xrightarrow{e_1^*(\sigma)} \cdots \xrightarrow{e_{n-1}^*(\sigma)} x_n$$

of $N(\mathcal{C})_n$.

To see that this $n$-simplex maps to $\sigma$, we need to check agreement on all 1- and 2-dimensional faces; this is left to the reader (hint: use that 2-simplices enforce a composition). □

# Part II

# $\infty$-groupoids

# Chapter 4

# The definition

## 4.1 Introduction

At the start of these notes, we explained that we are studying simplicial sets because of how they are helpful in modelling higher groupoids.

In the previous chapter, we discovered why: there is a fully faithful functor

$$N : \mathrm{Cat} \to \mathrm{sSet}$$

from the category of categories into the category of simplicial sets. In particular, the category of groupoids embeds fully faithfully into the category of simplicial sets. In this chapter, we study this embedding more closely, and then *define* $\infty$-groupoids as a certain full subcategory of sSet .

## 4.2 The essential image of $N$

We wish to define $\infty$-groupoids as a full subcategory of sSet . But which full subcategory? To get our first hint, we describe the essential image of $N$ intrinsically in sSet .

### 4.2.1 Categories

**Ideation 4.2.1.** What can we say about simplicial sets of the form $N(\mathcal{C})$, for a category $\mathcal{C}$? Observe that a 1-truncated simplicial set is *almost* a category: more precisely, a 1-truncated simplicial set consists of two sets $A, X$ together with maps

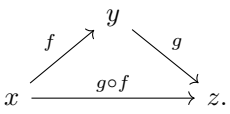
$$d^0, d^1 : A \to X,$$

$$s : X \to A.$$

We can interpret $A$ as a set of morphisms, $X$ as a set of objects, $d^1, d^0$ as the source and target of a morphism, and $s$ as a map sending an object to its identity morphism. This looks like a category, with one exception: we have no composition structure.

The nerve $N(\mathcal{C})$ uses the higher simplices to encode the composition structure on $\mathcal{C}$. So, to solve our recognition problem, we will determine when a simplicial set $X_\bullet$ is secretly encoding a composition structure on its 1-simplices.

To start, we look at 2-simplices. In $N(\mathcal{C})_2$, all 2-simplices look like




In other words, recalling the horn $\Lambda^1_2$ from Definition 2.3.9, we find that *every map* $\Lambda^1_2 \to N(\mathcal{C})$ *extends uniquely to* $\Delta_2$. A map $\Lambda^1_2 \to N(\mathcal{C})$ just encodes a composable pair of morphisms, and the extension to $\Delta_2$ records the composite map.

This is a good start, but what about higher simplices? The 3-simplices in $N(\mathcal{C})$ are just composable triples

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w.$$

In other words, if we look at the simplicial subset $A_3$ of $\Delta_3$ spanned by the edges $01, 12, 23$, then every map $A_3 \to N(\mathcal{C})$ extends *uniquely* to a map $\Delta_3 \to N(\mathcal{C})$.

Similarly, we can define $A_n \subseteq \Delta_n$ the (1-dimensional) simplicial subset spanned by the edges $01, 12, ..., (n-1)n$. An $n$-simplex of $N(\mathcal{C})$ should be uniquely determined by its restriction to this 1-dimensional subset.

**Theorem 4.2.2.** *Let $A_n \subseteq \Delta_n$ denote the simplicial subset spanned by the edges $01, 12, ..., (n-1)n$. Then a simplicial set $X$ is isomorphic to the nerve of a category if, and only if, the map*

$$\mathrm{Mor}(\Delta_n, X) \to \mathrm{Mor}(A_n, X)$$

*is a bijection for all $n \geq 2$.*

*Proof.* If $X = N(\mathcal{C})$, this claim is immediate from the description of $N(\mathcal{C})_n$ in terms of composable tuples.

Suppose now that $X$ satisfies this property. Define a category $\mathcal{C}$ whose set of objects is $X_0$, and where the set of morphisms from $x$ to $y$ is just the set of edges in $X_1$ with source $x$ and target $y$.

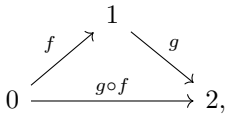
We give $\mathcal{C}$ the following composition structure: for $f : x \to y, g : y \to z$, let $g \circ f$ be the boundary edge opposite 1 in the unique map $\Delta_2 \to X$ which extends the map $\Lambda^1_2 \to X$ encoding the pair $f, g$.

We must prove that this composition is associative and unital. Degenerate edges function as units, with degenerate 2-simplices proving that the degenerate edges have the desired composition properties.

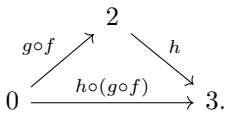
For associativity, let $f : x \to y, g : y \to z, h : z \to w$ be three composable morphisms. They define a map $(f, g, h) : A_3 \to X$, which by hypothesis extends uniquely to a 3-simplex $\sigma$ of $X$.

We wish to prove that $h \circ (g \circ f) = (h \circ g) \circ f$. This will follow from staring at each face of $\sigma$.

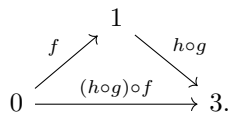
Indeed, the 012 face is




and so the 023 face is



But the 013 face of $\sigma$ is




In other words, the 013 face proves that $\sigma$ has 03 edge $(h \circ g) \circ f$, and the 023 face proves that $\sigma$ has 03 edge $h \circ (g \circ f)$. Thus these two compositions must be equal, proving associativity, as desired.

To see $N(\mathcal{C}) = X$, just compose the natural bijections

$$X_n \simeq \mathrm{Mor}(A_n, X),$$

$$\mathrm{Mor}(A_n, X) \simeq N(\mathcal{C})_n.$$

□

There is a fun way to rewrite this condition, found by Grothendieck (but named after Segal, who used this characterization in an influential 1968 paper). Theorem 4.2.2 tells us that, in the nerve of a category, simplices are determined by $n$ 'composable' edges.

**Definition 4.2.3** (Segal condition)**.** A simplicial set $X$ is said to obey the *Segal condition* if, for every $n \geq 2$, the map

$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is a bijection, where

1. in the fiber products, if $X_0$ is to the right of $X_1$, then $X_1 \to X_0$ is the target map, and if $X_0$ is to the left, then $X_1 \to X_0$ is the source map, and there are $n$ total copies of $X_1$;
2. the $i^{\text{th}}$ projection $X_n \to X_1$ in the above is just the restriction of an $n$-simplex to its $(i-1)i$ edge.

**Corollary 4.2.4.** *A simplicial set $X$ is the nerve of a category if and only if it obeys the Segal condition.*

*Proof.* As $A_n$ is obtained by attaching $n$ edges at their endpoints sequentially, we find

$$X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \simeq \mathrm{Mor}(A_n, X).$$

□

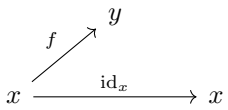
### 4.2.2 Groupoids

Theorem 4.2.2 detects which simplicial sets are categories, but at the moment we are most interested in groupoids. Thus we seek a modification of Theorem 4.2.2 which detects groupoids.

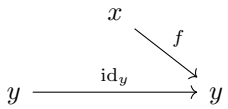
**Ideation 4.2.5.** How do we tell that $\mathcal{G}$ is a groupoid from looking at the simplicial set $N(\mathcal{G})$?

The sole difference between a groupoid and an arbitrary category is that a groupoid is required to have inverses. This means that, for any morphism $f : x \to y$, we can find another morphism $f^{-1} : y \to x$ so that $f^{-1} \circ f = \mathrm{id}_x$ and $f \circ f^{-1} = \mathrm{id}_y$.

In other words, in $N(\mathcal{G})$, we can fill both




and




to 2-simplices, having the same extra side. Actually, this 'same extra side condition' is forced on us by associativity, so we only need to require that both of those diagrams fill into 2-simplices.

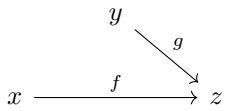
Thus, $N(\mathcal{G})$ has the additional property that any map

$$\Lambda_2^0 \to N(\mathcal{G})$$

or

$$\Lambda_2^2 \to N(\mathcal{G})$$

whose 02-side is degenerate can be extended to $\Delta_2$. This condition that the 1-side is degenerate is a bit awkward to state; however, as we should also be able to fill in a horn like




using $g^{-1} \circ f$ as the $xy$ side, this degenerate side condition is not needed. So, to get a cleaner statement, we won't say it.

**Theorem 4.2.6.** *A simplicial set $X$ is the nerve of a groupoid if and only if it is the nerve of a category, and every map $\Lambda_2^i \to X$ (for any $0 \leq i \leq 2$) can be extended uniquely to a map $\Delta_2 \to X$.*

*Proof.* Immediate from Ideation 4.2.5. □

Due to the appearance of all three 2-horns in this statement, for symmetry reasons people tend to state these recognition theorems as follows.

**Corollary 4.2.7.** *Let $X$ be a simplicial set. Then $X$ is the nerve of a category if and only if, for every $n \geq 2$ and $0 < i < n$, the map*

$$\mathrm{Mor}(\Delta_n, X) \to \mathrm{Mor}(\Lambda_n^i, X)$$

*is a bijection.*

*Moreover, $X$ is the nerve of a groupoid if and only if for every $n \geq 2$, and every $0 \leq i \leq n$, the map*

$$\mathrm{Mor}(\Delta_n, X) \to \mathrm{Mor}(\Lambda_n^i, X)$$

*is a bijection.*

*Proof.* We leave this as an exercise to the reader; the argument is quite similar to the proof of Theorem 4.2.2. □

## 4.3 Strict 2-groupoids as simplicial sets

How might we embed strict 2-groupoids into sSet?

**Ideation 4.3.1.** The categorical nerve (which we used to embed groupoids into sSet) was defined as the singular functor associated to a certain cosimplicial category. While the poset categories $[n]$ are not groupoids (and hence cannot naturally be viewed as strict 2-groupoids), we could still try copying this definition: in other words, for a strict 2-groupoid

$$\mathcal{G}_\bullet = \mathcal{G}_1 \rightrightarrows \mathcal{G}_0,$$

we could hope to define the nerve by saying its $n$-simplices are functors $[n] \to \mathcal{G}_\bullet$.

But what is the correct definition of a functor from a 1-category (like $[n]$) to a strict 2-groupoid? One obvious definition is to say that a functor $[n] \to \mathcal{G}_\bullet$ consists of objects $x_0, ..., x_n$, together with objects $e_{ij} \in \mathcal{G}_1$ with $s(e_{ij}) = x_i, t(e_{ij}) = x_j$, and obeying $\mu(e_{ij}, e_{jk}) = e_{ik}, e_{ii} = e(x_i)$ (where $\mu$ was our composition and $e : \mathcal{G}_0 \to \mathcal{G}_1$ was the identity section).

Unfortunately, this definition doesn't interact with the morphisms in $\mathcal{G}_1$ at all, and so it's not a great notion; if we tried using this notion of functor to define the nerve of a 2-groupoid, the resulting simplicial set wouldn't see the morphisms in $\mathcal{G}_1$ at all.

Can we modify the notion of functor, so that it sees the higher morphisms of our 2-groupoid?

One observation is that $\mu(e_{ij}, e_{jk})$ and $e_{ik}$ are both objects of $\mathcal{G}_1$. In a groupoid like $\mathcal{G}_1$, you could demand two objects are equal – like we did above – or you could instead demand that they are *isomorphic*.

So, let's try changing the definition of functor: instead of asking $\mu(e_{ij}, e_{jk}) = e_{ik}$, let's only ask for a choice of isomorphism $\mu(e_{ij}, e_{jk}) \simeq e_{ik}$ in $\mathcal{G}_1$. One could also loosen the identity requirement – asking merely that you provide an isomorphism $e_{ii} \simeq e(x_i)$ instead – but let's only loosen one constraint at a time.

At the level of [0] or [1] this changes nothing. But, under this weaker definition, there are now more functors $[2] \to \mathcal{G}_\bullet$.

Indeed, we now need to encode three objects $x_0, x_1, x_2 \in \mathcal{G}_0$, together with three objects $e_{01}, e_{02}, e_{12}$, and an isomorphism

$$\alpha : \mu(e_{01}, e_{12}) \simeq e_{02}$$

in $\mathcal{G}_1$.

This is great! The functors $[2] \to \mathcal{G}_\bullet$ now can encode the morphisms of $\mathcal{G}_1$!

However, there is one problem when we go to [3]. A functor $[3] \to \mathcal{G}_\bullet$ should include isomorphisms

$$\alpha : \mu(e_{01}, e_{12}) \simeq e_{02},$$

$$\beta : \mu(e_{01}, e_{13}) \simeq e_{03},$$

$$\gamma : \mu(e_{12}, e_{23}) \simeq e_{13},$$

and

$$\delta : \mu(e_{02}, e_{23}) \simeq e_{03}.$$

But note that $\alpha, \beta, \gamma$ induce the isomorphism

$$\mu(e_{02}, e_{23}) \xrightarrow{\alpha^{-1}} \mu(\mu(e_{01}, e_{12}), e_{23}) = \mu(e_{01}, \mu(e_{12}, e_{23})) \xrightarrow{\gamma} \mu(e_{01}, e_{13}) \xrightarrow{\beta} e_{03}.$$

So, we have two different isomorphisms

$$\mu(e_{02}, e_{23}) \simeq e_{03},$$

one given by $\delta$, and one built from $\alpha, \beta$, and $\gamma$. Should we require them to be related?

To get a good notion of functor, we really do want these isomorphisms to be related. To see why, think about how one uses the functor property; in practice, you use it to rewrite

$$F(g \circ f) = F(g) \circ F(f)$$

in the middle of some computation. The above example is a special case of a slightly more general phenomenon: whenever we try rewriting

$$F(h \circ g \circ f),$$

there are two ways to rewrite it, depending on how we group terms: we could do

$$F(h \circ g \circ f) = F(h) \circ F(g \circ f) = F(h) \circ F(g) \circ F(f),$$

or

$$F(h \circ g \circ f) = F(h \circ g) \circ F(f) = F(h) \circ F(g) \circ F(f).$$

When dealing with usual functors, these are just two different ways to prove that two morphisms in the target category are equal. But with these new functors, this gives us two different isomorphisms between $F(h \circ g \circ f)$ and $F(h) \circ F(g) \circ F(f)$.

It would be very, very annoying to have to pay that close of attention to the way you group arrows together in the middle of a lengthy calculation with functors. So, people typically require that those two isomorphisms coincide, leading to the notion of a *normal lax functor*, which we define formally below.

**Definition 4.3.2** (Normal lax functor)**.** Let $\mathcal{C}$ be a 1-category, and $\mathcal{G}_\bullet = (\mathcal{G}_0, \mathcal{G}_1, s, t, e, \mu)$ a strict 2-groupoid.

A *normal lax functor* $F : \mathcal{C} \to \mathcal{G}_\bullet$ is the data of

1. for each object $x \in \mathcal{C}$, an object $F(x) \in \mathcal{G}_0$,
2. for each morphism $f \in \mathcal{C}$, an object $F(f) \in \mathcal{G}_1$, obeying $s(F(f)) = F(s(f)), t(F(f)) = F(t(f))$,
3. for each pair of composable morphisms $f : x \to y, g : y \to z$ in $\mathcal{C}$, an isomorphism

$$\alpha_{g,f} : F(g \circ f) \to F(g) \circ F(f)$$

in $\mathcal{G}_1$,

such that

1. $F(\mathrm{id}_x) = e(F(x))$,
2. $\alpha_{f,\mathrm{id}} = \alpha_{\mathrm{id},f} = \mathrm{id}$,

3. for any composable triple $f : x \to y, g : y \to z, h : z \to w$, the two isomorphisms

$$F(h \circ g \circ f) \xrightarrow{\alpha_{h,g\circ f}} F(h) \circ F(g \circ f) \xrightarrow{\alpha_{g,f}} F(h) \circ F(g) \circ F(f),$$

$$F(h \circ g \circ f) \xrightarrow{\alpha_{h\circ g,f}} F(h \circ g) \circ F(f) \xrightarrow{\alpha_{h,g}} F(h) \circ F(g) \circ F(f)$$

are equal (as morphisms in $\mathcal{G}_1$).

**Remark 4.3.3.** A *lax functor* is one where we replace the strict identity requirement by an isomorphism. In the opposite direction, a *strict functor* is one in which we require the isomorphisms $\alpha$ to be the identity.

**Remark 4.3.4.** Why are we using normal lax functors instead of lax functors?

At the moment, we are trying to embed strict 2-groupoids into simplicial sets. The easiest way to do that seems to be to use normal lax functors, as they're a little simpler than lax functors. And, as we saw in Ideation 4.3.1, normal lax functors allow us to encode all the parts of a strict 2-groupoid, so they seem good enough.

A second reason is that simplicial sets have degeneracy morphisms which, in the case of the nerve of an ordinary category, encode the identity morphisms. As simplicial set maps are strictly compatible with degeneracies, it seems that if we're going to embed 2-groupoids into simplicial sets, we should only use functors strictly compatible with identity.

**Definition 4.3.5.** A *normal lax functor* between two strict 2-groupoids

$$F : \mathcal{G}_\bullet \to \mathcal{G}'_\bullet$$

is the data of

1. a functor $F_0 : \mathcal{G}_0 \to \mathcal{G}'_0$,
2. a functor $F_1 : \mathcal{G}_1 \to \mathcal{G}'_1$,
3. a natural transformation $\alpha$ from

$$(\mathcal{G}_1, t) \times_{\mathcal{G}_0} (\mathcal{G}_1, s) \xrightarrow{\mu} \mathcal{G}_1 \xrightarrow{F_1} \mathcal{G}'_1$$

to

$$(\mathcal{G}_1, t) \times_{\mathcal{G}_0} (\mathcal{G}_1, s) \xrightarrow{(F_1,F_1)} (\mathcal{G}'_1, t') \times_{\mathcal{G}'_0} (\mathcal{G}'_1, s') \xrightarrow{\mu'} \mathcal{G}'_1$$

such that

1. $s' \circ F_1 = F_0 \circ s$,
2. $t' \circ F_1 = F_0 \circ t$,
3. $F_1 \circ e = e' \circ F_0$,
4. $\alpha_{f,\mathrm{id}} = \mathrm{id}$,
5. $\alpha_{\mathrm{id},f} = \mathrm{id}$,
6. for $f : x \to y, g : y \to z, h : z \to w$ in $\mathcal{G}_1$, the diagram

$$\begin{array}{ccc} F_1(h\circ g\circ f) & \xrightarrow{\alpha_{h,g\circ f}} & F_1(h)\circ F_1(g\circ f) \\ \downarrow{\scriptstyle \alpha_{h\circ g,f}} & & \downarrow{\scriptstyle \alpha_{g,f}} \\ F_1(h\circ g)\circ F_1(f) & \xrightarrow{\alpha_{h,g}} & F_1(h)\circ F_1(g)\circ F_1(f) \end{array}$$

commutes.

**Construction 4.3.6** (The Duskin nerve of a strict 2-groupoid)**.** Let $2\,\mathrm{Gpd}_{\mathrm{nlax}}$ denote the category of strict 2-groupoids with normal lax functors as morphisms. We define the *Duskin nerve*

$$N^D : 2\,\mathrm{Gpd}_{\mathrm{nlax}} \to \mathrm{sSet}$$

as

$$N^D(\mathcal{G}_\bullet)_n := \mathrm{Fun}_{\mathrm{nlax}}([n], \mathcal{G}_\bullet).$$

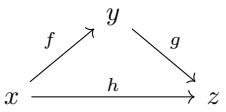
**Remark 4.3.7.** Let

$$\mathcal{G}_1 \rightrightarrows X$$

be a strict 2-groupoid (so $X$ is a set).

Then

1. 0-simplices of $N^D(\mathcal{G}_\bullet)$ are points of $X$,
2. 1-simplices are objects of $\mathcal{G}_1$,
3. a 2-simplex with boundary




is an isomorphism $\alpha : h \xrightarrow{\sim} \mu(f,g)$ in $\mathcal{G}_1$.

A 3-simplex is the same as a normal lax functor $[3] \to \mathcal{G}_\bullet$, or in other words:

1. objects $x_0, x_1, x_2, x_3 \in X$,
2. objects $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23} \in \mathcal{G}_1$, with $s(f_{ij}) = x_i, t(f_{ij}) = x_j$,
3. isomorphisms $\alpha_{ijk} : f_{ik} \xrightarrow{\sim} \mu(f_{ij}, f_{jk})$ in $\mathcal{G}_1$,

such that the isomorphisms obey the associativity constraint. In this case, the only non-trivial associativity constraint comes from $f_{23} \circ f_{12} \circ f_{01}$. The only consequence of this constraint is that any of the four isomorphisms $\alpha$ can be written in terms of the other 3; so we need only encode 3 isomorphisms instead of 4. Let's say we encode $012, 123$, and $023$.

In particular, for any $0 \le i \le 3$, the map

$$\mathrm{Mor}(\Delta_3, N^D(\mathcal{G}_\bullet)) \to \mathrm{Mor}(\Lambda^i_3, N^D(\mathcal{G}_\bullet))$$

is a bijection: maps out of $\Lambda^i_3$ encode the same information but with a "missing isomorphism" $\alpha$ (on face opposite vertex $i$), but the associativity constraint says that this missing $\alpha$ can always be uniquely found.

**Remark 4.3.8.** It is true that the Duskin nerve is fully faithful, but we will not need or prove this.

## 4.4 Embedding $\infty$-groupoids into simplicial sets

**Ideation 4.4.1.** In Remark 4.3.7, we observed that, for the Duskin nerve of a strict 2-groupoid, each map

$$\mathrm{Mor}(\Delta_3, N^D(\mathcal{G}_\bullet)) \to \mathrm{Mor}(\Lambda^i_3, N^D(\mathcal{G}_\bullet))$$

is a bijection.

Similarly, in Ideation 4.2.5, we observed that, for the nerve of a groupoid,

$$\mathrm{Mor}(\Delta_2, N(\mathcal{G})) \to \mathrm{Mor}(\Lambda^i_2, N(\mathcal{G}))$$

is always a bijection. This bijectivity fails for $N^D(\mathcal{G}_\bullet)$ because the 2-morphisms create extra $\Delta_2$'s which don't come from compositions; however, *compositions and inverses still always exist*, so

$$\mathrm{Mor}(\Delta_2, N^D(\mathcal{G})) \to \mathrm{Mor}(\Lambda^i_2, N^D(\mathcal{G}))$$

is still always surjective, even though we can lose injectivity thanks to 2-morphisms.

Going one level lower, we can view any set as a 0-dimensional simplicial set; then the map

$$\mathrm{Mor}(\Delta_1, X) \to \mathrm{Mor}(\Lambda^i_1, X)$$

is always a bijection: all edges are degenerate and hence determined by either one of their vertices. For groupoids, though,

$$\mathrm{Mor}(\Delta_1, N(\mathcal{G})) \to \mathrm{Mor}(\Lambda^i_1, N(\mathcal{G}))$$

is in general only *surjective*: identity morphisms show that every object is the source/target of some morphism, but a given object might be the source or target of *many* morphisms.

These three examples (strict 2-groupoids, groupoids, and sets) suggest the following: for every $k \geq n+1$, the maps

$$\mathrm{Mor}(\Delta_k, \text{nerve of } n\text{-groupoid}) \to \mathrm{Mor}(\Lambda^i_k, \text{nerve of } n\text{-groupoid})$$

should be bijections, for every $1 \leq k \leq n$, the maps

$$\mathrm{Mor}(\Delta_k, \text{nerve of } n\text{-groupoid}) \to \mathrm{Mor}(\Lambda^i_k, \text{nerve of } n\text{-groupoid})$$

should be *surjections*.

Setting $n = \infty$, we get the following definition.

**Definition 4.4.2.** An *$\infty$-groupoid* is a simplicial set $X_\bullet$ such that, for every $n \geq 1$ and every $0 \leq i \leq n$, the map

$$\mathrm{Mor}(\Delta_n, X) \to \mathrm{Mor}(\Lambda^i_n, X)$$

is a surjection.

**Remark 4.4.3.** People also use the term *Kan complex* for what we call $\infty$-groupoids. We will usually take the following convention:

1. when we wish to emphasize that we are thinking of $X_\bullet$ as a simplicial set, we use the term Kan complex;

2. when we wish to emphasize that we are thinking of $X_\bullet$ as a higher groupoid, we use the term $\infty$-groupoid.

Inspired by how we embedded categories into sSet, we make the following definition.

**Definition 4.4.4.** Let $X$ be an $\infty$-groupoid. An *object* of $X$ is an element of $X_0$. A *1-morphism* of $X$ is an element of $X_1$; we write a 1-morphism as $a : x \to y$, where $a \in X_1$, and $x = d^1(a), y = d^0(a)$.

The *identity morphism* associated to $x \in X_0$ is the morphism $s^0(x) \in X_1$.

We say that $c : x \to z$ is a *composition* of $a : x \to y$ and $b : y \to z$ if there is a 2-simplex $\sigma \in X_2$ whose 01 side is $a$, whose 12 side is $b$, and whose 02 side is $c$.

**Warning 4.4.5.** Unlike in 1-groupoids, in $\infty$-groupoids compositions are not unique! We can already see this at the level of strict 2-groupoids; intuitively, one should think that the 2-simplex $\sigma$ above witnesses an *isomorphism* $b \circ a \simeq c$. However, in part because it's bad to distinguish between isomorphic objects and in part because it sounds better, we prefer to say $c$ is a composition of $b$ and $a$ instead of saying $c$ is isomorphic to the composition of $b$ and $a$.

In Chapter 6, we will study 1-morphisms in more detail, and also define the $n$-morphisms of an $\infty$-groupoid.

## 4.5 Equivalence of $\infty$-groupoids

We know that

$$N : \mathrm{Cat} \to \mathrm{sSet}$$

is fully faithful, and we have characterized its essential image. However, knowing $N$ is fully faithful isn't everything: Cat, in addition to having objects and morphisms, also contains *natural transformations*: morphisms between functors. How does $N$ interact with them? This will lead us to a notion of natural transformations between functors between $\infty$-groupoids.

**Ideation 4.5.1.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be two parallel functors, and $\eta : F \to G$ a natural transformation.

Recall that a natural transformation is defined by a slightly funny set of data: for every *object* of $\mathcal{C}$, the natural transformation gives us a *morphism* of $\mathcal{D}$. Thinking in this way, we could even *define* a natural transformation to be a functor

$$\eta : \mathcal{C} \to \mathrm{Fun}([1], \mathcal{D}),$$

where $\Delta_1 = N([1])$ is the category on two objects, with one morphism between them, and so $\mathrm{Fun}([1], \mathcal{D})$ is the category of morphisms of $\mathcal{D}$. By currying, this data is equivalent to the data of a functor

$$\mathcal{C} \times [1] \to \mathcal{D}.$$

This leads to the following definition.

**Definition 4.5.2.** A *homotopy* from $f : X \to Y$ to $g : X \to Y$ of simplicial sets is a map

$$H : X \times \Delta_1 \to Y$$

such that $H|_{X \times \{0\}} = f, H|_{X \times \{1\}} = g$.

**Definition 4.5.3.** We say that $f, g : X \to Y$ are *naively homotopic* if there is a homotopy from $f$ to $g$. We say $f, g : X \to Y$ are *homotopic* if they belong to the same equivalence class of the equivalence relation generated by naive homotopy.

**Notation 4.5.4.** We often write $f \simeq g$ to denote that $f, g$ are homotopic.

**Warning 4.5.5.** Naive homotopy is *not* an equivalence relation in general! When $Y$ is an $\infty$-groupoid, though, naive homotopy of maps $X \to Y$ *is* an equivalence relation; we will show this in Corollary 4.7.2.

Homotopy of maps leads naturally to the following notion.

**Definition 4.5.6.** Two simplicial sets $X, Y$ are *homotopy equivalent* if there exist maps $f : X \to Y, g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$.

**Notation 4.5.7.** We write $X \simeq Y$ if $X, Y$ are homotopy equivalent.

**Warning 4.5.8.** As with Warning 4.5.5, one should beware that if $X, Y$ are not $\infty$-groupoids, then this notion of homotopy equivalence is not very well-behaved.

Homotopy is to $\infty$-groupoids as natural transformation is to ordinary groupoids; as such, homotopy will be an incredibly important concept in our work below.

In ordinary category theory, the existence of natural transformations is equivalent to the fact that the set of functors between two categories themselves forms a category. Similarly, one can turn the set of morphisms between two simplicial sets into a simplicial set of morphisms.

### 4.5.1 Mapping spaces

**Construction 4.5.9** (Mapping spaces)**.** Let $X, Y \in \mathrm{sSet}$.

We can define a simplicial set $[X, Y] \in \mathrm{sSet}$ as

$$[X, Y]_n := \mathrm{Mor}_{\mathrm{sSet}}(X \times \Delta_n, Y).$$

We let $\varphi : [m] \to [n]$ induce the arrow

$$\Phi : [X, Y]_n \to [X, Y]_m$$

sending $X \times \Delta_n \to Y$ to

$$X \times \Delta_m \to X \times \Delta_n \to Y$$

where the map $X \times \Delta_m \to X \times \Delta_n$ is the map $\mathrm{id}_X \times \varphi$, recalling the Yoneda lemma

$$\mathrm{Mor}(\Delta_m, \Delta_n) = \mathrm{Mor}([m], [n]).$$

**Remark 4.5.10.** This definition is inspired by adjunction: we ought to have that

$$\mathrm{Mor}(\Delta_n, [X, Y]) = \mathrm{Mor}(X \times \Delta_n, Y).$$

**Notation 4.5.11.** We use the notations $[X, Y], \mathrm{Map}(X, Y)$, and $\mathrm{Fun}(X, Y)$ interchangeably for the simplicial set of morphisms between two simplicial objects, depending on when we want to think of $[X, Y]$ as a space of maps, or as an $\infty$-groupoid of functors.

We will use $\mathrm{Mor}_{\mathrm{sSet}}(X, Y)$ to denote the ordinary set of morphisms. In particular, $\mathrm{Mor}_{\mathrm{sSet}}(X, Y)$ is a set, but $[X, Y]$ is a simplicial set.

Remark 4.5.10 can in fact be made precise.

**Definition 4.5.12.** Let $\mathcal{A}$ be a category with finite coproducts. We can define a functor

$$\mathrm{Simp}(\mathcal{A}) \times \mathrm{sSet}^f \to \mathrm{Simp}(\mathcal{A}),$$

where $\mathrm{sSet}^f$ is the category of simplicial sets $S_\bullet$ where each $S_n$ is finite, as sending $(X_\bullet, S_\bullet)$ to the simplicial $\mathcal{A}$-object

$$(X_\bullet \times S_\bullet)_n := \coprod_{s \in S_n} X_n.$$

**Theorem 4.5.13.** *For any $X, Y, S \in \mathrm{sSet}$, we have an adjunction*

$$\mathrm{Mor}_{\mathrm{sSet}}(S, [X, Y]) = \mathrm{Mor}_{\mathrm{sSet}}(S \times X, Y).$$

*In fact, we even have the stronger statement that there is a natural isomorphism of simplicial sets*

$$[S, [X, Y]] \simeq [S \times X, Y].$$

*Proof.* The stronger statement implies the weaker one by mapping out of $\Delta_0$. The a priori weaker statement in fact even implies the stronger one; if the hom set statement holds, then for any $n$ we have

$$\begin{aligned} [S, [X, Y]]_n &= \mathrm{Mor}(\Delta_n \times S, [X, Y]) \\ &= \mathrm{Mor}(\Delta_n \times S \times X, Y) \\ &= [S \times X, Y]_n, \end{aligned}$$

and so we conclude.

Thus we need only prove this weaker statement. To prove it, we explicitly produce a unit-counit adjunction. Let the counit

$$\epsilon(Z) : Y \times [Y, Z] \to Z$$

be the map

$$\epsilon(Z)_n : Y_n \times \mathrm{Mor}(Y \times \Delta_n, Z) \to Z_n$$

given as follows. A map $\psi : Y \times \Delta_n \to Z$ determines a map $\psi_n : Y_n \times (\Delta_n)_n \to Z_n$. Let $*$ be the one nondegenerate $n$-simplex of $\Delta_n$, and put

$$\epsilon(Z)_n(y_n, \psi) = \psi_n(y_n, *).$$

Let the unit

$$\eta(X) : X \to [Y, X \times Y]$$

be given as follows. We need to give maps

$$\eta(X)_n : X_n \to \mathrm{Mor}(\Delta_n \times Y, X \times Y).$$

Define

$$\eta(X)_n(x_n) = x_n \times \mathrm{id}_Y,$$

viewing $x_n \in X_n$ as a map $x_n : \Delta_n \to X$.

These $\epsilon, \eta$ are clearly natural transformations. To check the unit counit equations, note that

$$\epsilon(X \times Y) \circ (\mathrm{id}_Y \times \eta(X))$$

sends $(y_n, x_n) \in Y_n \times X_n$ first to the pair

$$(y_n, x_n \times \mathrm{id}_Y),$$

and then to

$$(x_n \times \mathrm{id}_Y)(y_n) = (x_n, y_n).$$

So this composition is the identity. The other direction is similar. □

### 4.5.2 Homotopy via mapping spaces

We can rewrite the definition of homotopy in terms of path components of the mapping space.

**Proposition 4.5.14.** *Let $X, Y$ be two simplicial sets. Then $f, g : X \to Y$ are naively homotopic if and only if there is an edge in $[X, Y]$ from $f$ to $g$, and $f, g$ are homotopic if and only if they lie in the same path component of $[X, Y]$.*

*Proof.* Immediate from the tensor-hom adjunction. □

This perspective illustrates why homotopy is also a natural idea from the point of view of geometry. In fact, long before groupoids and categories were studied, geometers had studied homotopy, for a simple reason: oftentimes, one has *discretely valued invariants of continuous functions*. For example, the winding number of a continuous loop, or the degree of a smooth map of manifolds, are two $\mathbb{Z}$-valued invariants which depend continuously *on a function*. Discretely-valued continuous functions are constant on connected components, and hence studying them leads one to want to study connected components of spaces of functions – in other words, this leads to the study of homotopy.

Studying homotopy classes also has another purpose: they are a source of interesting isomorphism invariants of simplicial sets. For example, if $X \simeq Y$, then $\pi_0(X) \simeq \pi_0(Y)$. But the set of connected components is not a very strong invariant. Geometers, over time, observed that one could make more powerful invariants of the following form: fix an auxiliary simplicial set $T$; then $\pi_0([T, X])$ and $\pi_0([X, T])$ are also isomorphism invariants of $X$, and can often contain more information than $\pi_0(X)$ does.

### 4.5.3 Digression: Homotopies of continuous functions

This subsection is just a small elaboration on the remarks of the previous subsection, about the naturality of homotopy inside of topology.

**Definition 4.5.15.** Two continuous functions $f, g : X \to Y$ of topological spaces are *homotopic* if there is a continuous function $H : [0, 1] \times X \to Y$ such that $H(0, -) = f, H(1, -) = g$.

The theories of topological spaces and $\infty$-groupoids are very closely connected: every topological space $X$ gives rise to an $\infty$-groupoid, and homotopic maps of topological spaces give naturally isomorphic functors between the resulting groupoids.

**Theorem 4.5.16.** *For any topological space $X$, the simplicial set $\mathrm{Sing}(X)$ is actually a Kan complex.*

*Moreover, if $f, g : X \to Y$ are two homotopic maps, then $\mathrm{Sing}(f), \mathrm{Sing}(g)$ are homotopic.*

*Proof.* For the first part, let

$$r : \Delta^n \to |\Lambda^i_n|$$

be the continuous map

$$r(x_0, ..., x_n) = (x_0 - \varphi(x), ..., x_{i-1} - \varphi(x), x_i + n\varphi(x), x_{i+1} - \varphi(x), ..., x_n - \varphi(x)),$$

where

$$\varphi(x_0, ..., x_n) = \min_{j \neq i}\{x_j\}.$$

Then $r$ is a retraction, and so precomposing a continuous map $|\Lambda^i_n| \to X$ with $r$ gives a continuous map $|\Delta_n| \to X$ whose restriction to $|\Lambda^i_n|$ is our original map. Thus $\mathrm{Sing}(X)$ is a Kan complex.

For the second part, let $H : [0, 1] \times X \to Y$ be a homotopy from $f$ to $g$. Consider the composition

$$H' : \Delta_1 \times \mathrm{Sing}(X) \xrightarrow{a \times \mathrm{id}_{\mathrm{Sing}(X)}} \mathrm{Sing}(\Delta^1) \times \mathrm{Sing}(X) \simeq \mathrm{Sing}(\Delta^1 \times X) \xrightarrow{\mathrm{Sing}(H)} \mathrm{Sing}(Y),$$

where

1. the map $a : \Delta_1 \to \mathrm{Sing}(\Delta^1)$ is the one classifying the identity map $\Delta^1 \to \Delta^1$, and
2. the isomorphism $\mathrm{Sing}(\Delta^1) \times \mathrm{Sing}(X) \simeq \mathrm{Sing}(\Delta^1 \times X)$ comes from the fact that Sing, being a right adjoint, commutes with limits.

Then $H'$ is our desired homotopy, so we conclude. □

**Remark 4.5.17.** Theorem 4.5.16 is not complete; we would really like to prove some equivalence between the point-set notion of homotopy and the simplicial set notion of homotopy. This will be done in Chapter 13.

In Chapter 13, we will also prove that *every $\infty$-groupoid is equivalent to one of the form* $\mathrm{Sing}(X)$. Thus topological spaces up to homotopy provide another model of $\infty$-groupoids. In practice, it is sometimes convenient to use the topological space model, which is why we develop it.

**Remark 4.5.18** (Anima)**.** Clausen – inspired by Beilinson's usage of the word *animation* for lifting identities in $\pi_0(X)$ to specific paths in $X$ – introduced the term *anima* to refer to $\infty$-groupoids.

In this language, we think of an $\infty$-groupoid $X$ as an *animation* of the set $\pi_0(X)$, where each point of that discrete set is given an entire connected topological space to play in. We imagine each point of $\pi_0(X)$ as buzzing around in a playground.

In these notes, we will often use the term *anima* to mean $\infty$-groupoid.

Geometric realization sends homotopic maps to homotopic maps.

**Proposition 4.5.19.** *Let $f, g : X \to Y$ be two homotopic maps of simplicial sets. Then $|f|$ and $|g|$ are homotopic continuous functions.*

*Proof.* Immediate from Theorem 3.6.9. □

There is a variant of Proposition 4.5.19 which is also useful.

**Corollary 4.5.20.** *Let $S$ be a simplicial set and $X$ a point-set topological space. Assume that $f, g : S \to \mathrm{Sing}(X)$ are homotopic maps of simplicial sets. Then under the bijection*

$$\mathrm{Mor}(S, \mathrm{Sing}(X)) \simeq \mathrm{Mor}(|S|, X),$$

*the adjuncts $f', g' : |S| \to X$ are homotopic.*

*Proof.* By Proposition 4.5.19, the geometric realizations $|f|$ and $|g|$ are then homotopic. But the adjunct maps are obtained as the compositions

$$|S| \xrightarrow{|f|} |\,\mathrm{Sing}(X)| \xrightarrow{\epsilon_X} X,$$

$$|S| \xrightarrow{|g|} |\,\mathrm{Sing}(X)| \xrightarrow{\epsilon_X} X,$$

so that the homotopy from $|f|$ to $|g|$ can be postcomposed with $\epsilon_X$ to get a homotopy between the adjuncts. □

**Remark 4.5.21.** At the moment, we do not have any tools that would allow us to go from homotopies of continuous functions to homotopies of simplicial set maps. This will be fixed in Chapter 13.

## 4.6 The $\pi_0$ of an $\infty$-groupoid

When $X$ is an $\infty$-groupoid, we think of $\pi_0(X)$ as the set of isomorphism classes of objects in $X$. Fortunately, for $\infty$-groupoids, it is much easier to deal with $\pi_0(X)$ than in general.

**Proposition 4.6.1.** *Let $X$ be a Kan complex. Then naive path equivalence is already an equivalence relation on $X_0$.*

*Proof.* For reflexivity, use degenerate edges.

For symmetry, use horn filling: if $e$ is an edge from $x$ to $y$, then fill a $\Lambda^0_2$ horn whose 01 side is $e$ and whose 02 side is $s(x)$ to get an edge from $y$ to $x$.

For transitivity, use $\Lambda^1_2$ horn filling. □

The set $\pi_0(X)$ is also the best set-theoretic approximation to any $\infty$-groupoid, in the following sense.

**Proposition 4.6.2.** *The functors*

$$\pi_0 : \mathrm{sSet} \to \mathrm{Set},$$

$$i : \mathrm{Set} \to \mathrm{sSet}$$

*are adjoint (with $\pi_0$ the left adjoint), where $i$ is the functor sending a set to the associated 0-dimensional simplicial set (see Example 2.3.4).*

*Moreover, the functor $i$ factors through* Kan *(the full subcategory of* sSet *consisting of Kan complexes), and the resulting functors*

$$\pi_0 : \mathrm{Kan} \to \mathrm{Set},$$

$$i : \mathrm{Set} \to \mathrm{Kan}$$

*are again an adjoint pair.*

*Proof.* As Kan $\hookrightarrow$ sSet is fully faithful, the second set of claims is immediate from the first. For the first, take $X$ a simplicial set and $S$ a set. We wish to produce a natural bijection

$$\mathrm{Mor}_{\mathrm{sSet}}(X, i(S)) \simeq \mathrm{Mor}_{\mathrm{Set}}(\pi_0(X), S).$$

A morphism of simplicial sets $\varphi : X \to i(S)$ is the data of functions

$$\varphi_n : X_n \to S$$

obeying the simplicial identities; in particular, as $i(S)$ is the constant simplicial object, we must have

$$\varphi_n(d^i(\sigma)) = d^i(\varphi_{n+1}(\sigma)) = \varphi_{n+1}(\sigma).$$

In particular, $\varphi_1, \varphi_2, \varphi_3, ...$ are uniquely determined by $\varphi_0$.

Moreover, a function

$$\varphi_0 : X_0 \to S$$

can be extended to a simplicial set morphism only if, for every edge $e \in X_1$, we have

$$\varphi_0(d^0(e)) = \varphi_0(d^1(e)),$$

as otherwise we could not create $\varphi_1$. Thus, we have a natural injection

$$\mathrm{Mor}_{\mathrm{sSet}}(X, i(S)) \hookrightarrow \mathrm{Mor}_{\mathrm{Set}}(\pi_0(X), S),$$

$$\varphi \mapsto \varphi_0,$$

using that $\varphi_0$ determines $\varphi$ and must be constant on path components.

To see that this is surjective, take $\varphi_0 : X_0 \to S$ a function of sets which is constant on path components. For every $n \geq 1$, and $\sigma \in X_n$, we define

$$\varphi_n(\sigma) := \varphi_0(\sigma(0)),$$

where $\sigma(0) \in X_0$ is the 0 vertex of $\sigma$. As all vertices of a simplex lie in the same path component, it is easy to see that the resulting $\varphi_n$ are compatible with degeneracy and face operators, and hence give a morphism of simplicial sets. We conclude. □

## 4.7 Absolutely anodyne maps

Let $X, Y$ be two $\infty$-groupoids. We can define a simplicial set $[X, Y]$, of functors from $X$ to $Y$. We expect this simplicial set to be an $\infty$-groupoid itself, and indeed it is.

**Theorem 4.7.1.** *If $Y$ is a Kan complex, then for any simplicial set $X$, the object $[X, Y]$ is a Kan complex.*

**Corollary 4.7.2.** *If $Y$ is a Kan complex, then two maps $f, g : X \to Y$ are homotopic if and only if they are naively homotopic.*

*Proof.* Proposition 4.5.14 tells us that naive homotopy is naive path equivalence in $[X, Y]$, and homotopy is path equivalence in $[X, Y]$. As $[X, Y]$ is a Kan complex by Theorem 4.7.1, Proposition 4.6.1 tells us that naive path equivalence and path equivalence coincide. □

**Ideation 4.7.3.** Uncurrying the definition of Kan complex, Theorem 4.7.1 is equivalent to proving the following: for any simplicial set $X$, and any Kan complex $Y$, the map

$$\mathrm{Mor}(\Delta_n \times X, Y) \to \mathrm{Mor}(\Lambda^i_n \times X, Y)$$

is surjective.

This suggests we study all simplicial set maps $A \to B$ having the property that

$$\mathrm{Mor}(B, Y) \to \mathrm{Mor}(A, Y)$$

is a surjection for every Kan complex $Y$.

**Definition 4.7.4.** An *absolutely anodyne* morphism of simplicial sets is a simplicial set map $A \to B$ such that, for every Kan complex $T$, the map

$$\mathrm{Mor}(B, T) \to \mathrm{Mor}(A, T)$$

is surjective.

**Warning 4.7.5.** This is non-standard terminology.

### 4.7.1 Strongly anodyne morphisms

How do we produce examples of absolutely anodyne morphisms?

Clearly each horn inclusion $\Lambda^i_n \hookrightarrow \Delta_n$ is absolutely anodyne. Using these, we can build some more absolutely anodyne maps.

**Proposition 4.7.6** (Stability properties of absolutely anodyne maps)**.** *A composition of absolutely anodyne maps is absolutely anodyne.*

*In any pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ B & \xrightarrow{g} & B', \end{array}$$

*if $i$ is absolutely anodyne, then so is $i'$.*

*In a commutative diagram*

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_A}{\frown} & & \\ A & \xrightarrow{i_A} & A' & \xrightarrow{r_A} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{i_B} & B' & \xrightarrow{r_B} & B, \\ & & \underset{\mathrm{id}_B}{\smile} & & \end{array}$$

*where $f'$ is absolutely anodyne, the map $f$ is also absolutely anodyne.*

*Moreover, if*

$$\{f_i : A_i \to B_i\}_{i \in I}$$

*is an arbitrary collection of absolutely anodyne morphisms, then the coproduct*

$$f : \coprod_i A_i \to \coprod_i B_i$$

*is absolutely anodyne.*

**Remark 4.7.7.** We say absolutely anodyne maps are stable under composition, pushouts, and retracts to refer to the first three results of this proposition.

*Proof.* Fix $X$ an arbitrary Kan complex.

The first claim is immediate from the fact that a composition of surjections is a surjection.

For the second claim, suppose we have a map $a' : A' \to X$. As $i : A \to B$ is absolutely anodyne, the map $a' \circ f : A \to X$ extends to a map $b : B \to X$. The pair $(a', b)$ then defines a map $B' \to X$ extending $a'$, and so $i'$ is absolutely anodyne.

For the third claim, let $a : A \to X$ be some morphism. Then $a \circ r_A : A' \to X$ extends to a map $b' : B' \to X$, as $f'$ is absolutely anodyne. From the commutativity of our diagram, we easily deduce that $b' \circ i_B : B \to X$ extends $a$, and so $f$ is absolutely anodyne.

The last claim is immediate from the universal property of coproducts. □

There is one more operation absolutely anodyne maps are stable under: $\omega$-composition.

**Proposition 4.7.8.** *Let*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

*be an infinite chain of absolutely anodyne morphisms. Then the $\omega$-composition*

$$X_0 \to \operatorname{colim}_n X_n$$

*is absolutely anodyne.*

*Proof.* Given any map $a_0 : X_0 \to T$ for a Kan complex $T$, inductively extend it to $X_n$ using that $f_{n-1} : X_{n-1} \to X_n$ is absolutely anodyne; as these extensions are all compatible, we get an induced map $\operatorname{colim}_n X_n \to T$ extending $a_0$, as desired. □

**Definition 4.7.9.** The class of *strongly anodyne morphisms* is the smallest class of morphisms in sSet such that

1. every horn inclusion $\Lambda^i_n \hookrightarrow \Delta_n$ is strongly anodyne,
2. every isomorphism is strongly anodyne,
3. the composition of two strongly anodyne morphisms is strongly anodyne,
4. the $\omega$-composition of strongly anodyne morphisms is strongly anodyne,
5. pushouts and retracts of strongly anodyne morphisms (in the sense of Proposition 4.7.6) are strongly anodyne,
6. arbitrary coproducts of strongly anodyne morphisms are strongly anodyne.

**Remark 4.7.10.** The class of monomorphisms is closed under all the operations above, and so every strongly anodyne morphism is a monomorphism.

**Lemma 4.7.11.** *Every strongly anodyne morphism is absolutely anodyne.*

*Proof.* Immediate from Proposition 4.7.6 and Proposition 4.7.8. □

To prove Theorem 4.7.1, it is enough to prove the following.

**Lemma 4.7.12.** *Let $A$ be a simplicial set. Then*

$$\Lambda^i_n \times A \hookrightarrow \Delta_n \times A$$

*is strongly anodyne.*

We defer the proof of Lemma 4.7.12 to Appendix A; the argument is rather combinatorial, and is easier to present once we suitably *generalize* Lemma 4.7.12. The motivation for this generalization is presented in the next section, so we strongly encourage the reader to defer reading the appendix until the end of Section 4.8.

*Proof of Theorem 4.7.1.* Immediate from Lemma 4.7.12 and the universal property of mapping spaces. □

## 4.8 Kan fibrations

Grothendieck taught us that, in mathematics, it is almost always helpful to consider *relative* versions of definitions. As these notes study Kan complexes in quite a bit of detail, it is natural to make the following relative variant of the notion of Kan complex.

**Definition 4.8.1.** A *Kan fibration* is a map of simplicial sets $f : X \to Y$ such that in any commutative diagram

$$\begin{array}{ccc} \Lambda^i_n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta_n & \longrightarrow & Y \end{array}$$

with $n \geq 1$, the dashed arrow can be filled in.

**Remark 4.8.2.** A Kan complex is therefore a simplicial set $X$ so that the map $X \to \mathrm{pt}$ is a Kan fibration.

### 4.8.1 Anodyne morphisms

Inspired by the notion of absolutely anodyne morphisms, we make the following definition.

**Definition 4.8.3.** An *anodyne morphism* is a morphism $f : A \to B$ of simplicial sets so that, for any Kan fibration $g : X \to S$, and any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \downarrow f & \nearrow & \downarrow g \\ B & \xrightarrow{\bar{b}} & S, \end{array}$$

we can always fill in the dashed arrow.

**Warning 4.8.4.** Every anodyne morphism is absolutely anodyne, but the converse is false!

As an example,

$$\mathrm{pt} \hookrightarrow \mathrm{pt} \coprod \mathrm{pt}$$

is absolutely anodyne, but not anodyne; we leave it to the reader to construct an example proving this map is not anodyne (it is rather easy to do this using even 0-dimensional simplicial sets).

Anodyne morphisms obey the exact same stability properties as absolutely anodyne morphisms, for essentially the same reasons.

**Proposition 4.8.5.** *Anodyne morphisms are stable under compositions, $\omega$-compositions, pushouts, retracts, and arbitrary coproducts.*

*In particular, every strongly anodyne morphism is anodyne.*

*Proof.* The same arguments used in the absolutely anodyne case go through with almost no modification; we leave this as an exercise to the reader. □

### 4.8.2 A remarkable uniqueness

**Ideation 4.8.6.** Let $X$ be an $\infty$-groupoid. Then

$$\mathrm{Mor}(\Delta_1, X) \to \mathrm{Mor}(\{1\}, X) \tag{4.8.7}$$

is a surjection, but not always a bijection. However, by a miracle, it is a bijection on *homotopy classes*.

Observe that (4.8.7) is a function between sets; if we take $\infty$-groupoids seriously, it is more natural to instead consider the functor

$$\theta : [\Delta_1, X] \to [\{1\}, X] = X$$

of $\infty$-groupoids. The surjectivity of (4.8.7) implies that

$$\pi_0([\Delta_1, X]) \to \pi_0 X$$

is a surjection. But remarkably, this map on homotopy classes is also an injection.

To see this, suppose $e, f : \Delta_1 \to X$ are such that $d^0(e)$ and $d^0(f)$ are homotopic as maps $\mathrm{pt} \to X$. This means we can find some path $p \in X_1$ from $d^0(e)$ to $d^0(f)$. We can represent this data with the diagram

$$\begin{array}{ccc} d^0(e) & \xrightarrow{p} & d^0(f) \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle f} \\ d^1(e) & & d^1(f). \end{array}$$

We can think of this diagram as a map $A \to X$, where $A \subseteq \Delta_1^2$ is the simplicial subset spanned by the three edges shown in the above diagram. Equivalently,

$$A = (\{0,1\} \times \Delta_1) \coprod_{\{0,1\} \times \{1\}} (\Delta_1 \times \{1\}).$$

A homotopy from $e$ to $f$ is the same as a map $\Delta^2_1 \to X$ whose $(0,0) \to (0,1)$ edge is $e$ and whose $(1,0) \to (1,1)$ edge is $f$. Thus, to prove $e$ is homotopic to $f$, it suffices to show that $A \to \Delta^2_1$ is anodyne.

Fortunately, this is rather easy; one can prove it is strongly anodyne via a direct construction with pushouts (see Ideation A.5.2).

Ideation 4.8.6 suggests the following general result.

**Theorem 4.8.8.** *Let $i : A \hookrightarrow B$ be* strongly *anodyne, and $j : C \hookrightarrow D$ an arbitrary inclusion of simplicial sets. Then*

$$(A \times D) \coprod_{A \times C} (B \times C) \hookrightarrow B \times D$$

*is strongly anodyne.*

With Theorem 4.8.8, we can prove the following very general homotopy uniqueness result.

**Theorem 4.8.9.** *Let $i : A \hookrightarrow B$ be a strongly anodyne inclusion, and suppose $f : X \to S$ is a Kan fibration. Then, in any lifting problem*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\bar{b}} & S, \end{array}$$

*any two solutions are* homotopic*. Moreover, given any two solutions $b_0, b_1 : B \to X$ of the lifting property, we may find a homotopy*

$$H : B \times \Delta_1 \to X$$

*from $b_0$ to $b_1$ such that*

1. *$f \circ H$ is the constant homotopy from $\bar{b}$ to itself,*
2. *$H|_{A \times \Delta_1}$ is the constant homotopy from $a$ to itself.*

*Proof.* By Theorem 4.8.8, the map

$$(A \times \Delta_1) \coprod_{A \times \{0,1\}} (B \times \{0,1\}) \hookrightarrow B \times \Delta_1 \tag{4.8.10}$$

is anodyne. From here, the claim is obvious: constructing our desired $H$ is equivalent to solving a lifting problem against (4.8.10), which is possible as it is anodyne. □

This homotopy uniqueness is very useful in practice.

### 4.8.3 Exponentiation for Kan fibrations

Theorem 4.8.8 has another, very useful, application. To explain it, we first observe a situation in which morphisms of the form of the one in Theorem 4.8.8 arise.

**Ideation 4.8.11.** Suppose $f : X \to S$ is a Kan fibration, $i : A \hookrightarrow B$ is strongly anodyne, and $j : C \hookrightarrow D$ is an inclusion. Then Theorem 4.8.8 tells us we can solve all lifting problems of the form

$$\begin{array}{ccc} (A \times D) \coprod_{A \times C} (B \times C) & \xrightarrow{\varphi} & X \\ \downarrow & \nearrow & \downarrow f \\ B \times D & \xrightarrow{\psi} & S. \end{array}$$

A map

$$\varphi : (A \times D) \coprod_{A \times C} (B \times C) \to X$$

is the same as a pair of maps

$$\varphi_{AD} : A \times D \to X,$$
$$\varphi_{BC} : B \times C \to X$$

whose restrictions to $A \times C$ agree.

Currying, these are the same as maps

$$\varphi_1 : A \to [D, X],$$
$$\varphi_2 : B \to [C, X]$$

such that $(\varphi_2)|_A = j^* \circ \varphi_1$.

In particular, solving our original lifting problem is equivalent to solving the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\varphi_1} & [D, X] \\ \downarrow & & \downarrow \\ B & \xrightarrow{(\psi, \varphi_2)} & [D, S] \times_{[C,S]} [C, X]. \end{array}$$

Thus

$$\theta : [D, X] \to [D, S] \times_{[C,S]} [C, X]$$

is a Kan fibration!

Similarly, we can deduce that

$$\theta' : [B, X] \to [B, S] \times_{[A,S]} [A, X]$$

is a very strong kind of Kan fibration: we can solve a lifting problem of *any inclusion* against $\theta'$.

These two observations lead to the following theorems.

**Definition 4.8.12.** A *trivial Kan fibration* is a morphism $f : X \to S$ of simplicial sets such that *any* lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & & \downarrow f \\ B & \longrightarrow & S \end{array}$$

with $i$ an inclusion has a solution.

**Theorem 4.8.13** (Exponentiation for Kan fibrations)**.** *Let $f : X \to S$ be a Kan fibration of simplicial sets, and $i : A \hookrightarrow B$ a monomorphism of simplicial sets (that is, $i$ is injective on $n$-simplices for all $n$). Then*

$$\theta : [B, X] \to [B, S] \times_{[A,S]} [A, X]$$

*is a Kan fibration.*

*Moreover, if $i$ is* strongly anodyne*, then $\theta$ is a* trivial *Kan fibration.*

*Proof.* Immediate from Ideation 4.8.11. □

### 4.8.4 Lifting in families

Our current definition of Kan fibration is somewhat morally wrong, in the sense that it is too set-theoretic. That is, our current definition says that $f : X \to S$ is a Kan fibration if, for every anodyne morphism $i : A \to B$, the function

$$\mathrm{Mor}(B, X) \to \mathrm{Mor}(B, S) \times_{\mathrm{Mor}(A,S)} \mathrm{Mor}(A, X)$$

is surjective. Recalling that surjective functions are the same as functions admitting sections, it is natural to ask if

$$\theta : [B, X] \to [B, S] \times_{[A,S]} [A, X]$$

admits a section. And in fact, $\theta$ *does* admit a section; at the moment, though, we can only prove this for $i : A \hookrightarrow B$ *strongly* anodyne. Moreover, the homotopy uniqueness of lifts is reflected in the fact that this section is actually a *homotopy inverse* to $\theta$. This is true for arbitrary trivial Kan fibrations.

**Theorem 4.8.14.** *Let $f : X \to S$ be a trivial Kan fibration. Then $f$ admits a section $s : S \to X$, and any section is a homotopy inverse of $f$.*

*Proof.* Solve a lifting problem against $\emptyset \hookrightarrow S$ to get a section; solving a lifting problem against $X \times \{0, 1\} \hookrightarrow X \times \Delta_1$ shows that any section is a homotopy inverse. □

## 4.9 Homotopy of functors between $\infty$-groupoids

Thanks to Theorem 4.7.1, when $Y$ is an $\infty$-groupoid, for any simplicial set $X$, the mapping space $[X, Y]$ is also an $\infty$-groupoid. Thus, by Proposition 4.6.1, $\pi_0[X, Y]$ can be computed naively. This leads to the following result.

**Proposition 4.9.1.** *Let $Y$ be an anima. Then, for any simplicial set $X$, two maps $f, g : X \to Y$ are homotopic if and only if they are naively homotopic.*

*Proof.* Immediate from Proposition 4.5.14. □

**Remark 4.9.2.** In particular, if $f : X \to Y$ is a functor between anima, then $f$ is a homotopy equivalence if and only if there exists a functor $g : Y \to X$ such that $gf$ is naively homotopic to $\mathrm{id}_X$ and $fg$ is naively homotopic to $\mathrm{id}_Y$.

Homotopy interacts very well with composition.

**Proposition 4.9.3.** *Let $X, Y, Z$ be simplicial sets, with $Z$ an $\infty$-groupoid. Suppose we have maps $f, f' : X \to Y$ and $g, g' : Y \to Z$.*

*Assume $f$ is homotopic to $f'$, and $g$ is homotopic to $g'$. Then*

$$g' \circ f' \simeq g \circ f.$$

*Proof.* It suffices to do this in the case where $f$ is naively homotopic to $f'$, as homotopy of maps is an equivalence relation.

Thus we may choose homotopies

$$H_f : \Delta_1 \times X \to Y,$$

$$H_g : \Delta_1 \times Y \to Z$$

from $f$ to $f'$ and from $g$ to $g'$, respectively.

Observe that $g \circ H_f$ is then a homotopy from $g \circ f$ to $g \circ f'$, and $H_g \circ (\mathrm{id}_{\Delta_1} \times f')$ is a homotopy from $g \circ f'$ to $g' \circ f'$. As homotopy of maps into $Z$ is an equivalence relation, we deduce

$$g \circ f \simeq g' \circ f'$$

by transitivity. □

## 4.10 A homotopy Yoneda lemma

The Yoneda lemma allows one to detect when a map of $\infty$-groupoids is an *isomorphism*. However, the more important notion for $\infty$-groupoids is homotopy equivalence. Thus we give the following homotopical variant of the Yoneda lemma.

**Theorem 4.10.1** (Homotopy Yoneda lemma)**.** *Let $f : X \to Y$ be a functor of $\infty$-groupoids. The following are equivalent:*

1. *$f$ is a homotopy equivalence;*
2. *for every simplicial set $A$, the map*

   $$\pi_0[A, X] \to \pi_0[A, Y]$$

   *is a bijection;*
3. *for every $\infty$-groupoid $A$, the map*

   $$\pi_0[A, X] \to \pi_0[A, Y]$$

   *is a bijection;*
4. *for every $\infty$-groupoid $A$, the map*

   $$\pi_0[Y, A] \to \pi_0[X, A]$$

   *is a bijection.*

*Proof.* Assume first that $f$ is a homotopy equivalence; choose a homotopy inverse $g$.

If $fa \simeq fa'$, then $gfa \simeq gfa'$ and hence $a \simeq a'$ as $gf \simeq \mathrm{id}$. Thus $\pi_0[A, X] \to \pi_0[A, Y]$ is injective. For surjectivity, given any map $a : A \to Y$, observe that $f(ga) = (fg)a \simeq a$, and so condition 1 implies condition 2. The same argument shows condition 1 implies conditions 3 and 4 as well.

Condition 2 of course implies condition 3. Condition 3 allows us to construct a homotopy inverse to $f$: find some $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$. Using the injectivity part of condition 3, we see that $fgf \simeq f$ implies $gf \simeq \mathrm{id}_X$, so that $f, g$ are homotopy inverses.

A similar argument shows condition 4 implies condition 1, and hence we conclude. □

# Chapter 5

# Homotopy fibers

## 5.1 Introduction

In the previous chapter, we defined $\infty$-groupoids and functors between them. But these definitions are not enough, on their own, to use $\infty$-groupoids in practice; we really want to have $\infty$-groupoid versions of all the standard operations of set theory.

One of the most standard set-theoretic operations is to form the *fibers* of a function between sets. The $\infty$-groupoid analogue of this notion is called the *homotopy fiber* of a functor between anima. The goal of this chapter is to explore homotopy fibers.

## 5.2 Homotopy fibers

### 5.2.1 Strict fibers

The most obvious way to define the fiber of a functor between anima is as follows.

**Definition 5.2.1.** Let $f : X \to S$ be a morphism of simplicial sets. For each $s \in S_0$, the *strict fiber* of $f$ above $s$ is the simplicial set

$$X_s := \{s\} \times_S X.$$

Unfortunately, the strict fiber is rather poorly behaved from the point of view of $\infty$-groupoids. There are two main defects:

1. The strict fiber of a functor between $\infty$-groupoids is not always an $\infty$-groupoid itself.
2. The strict fiber depends *strictly* on $s$; to have a well-behaved notion, one should want some fiber which depends only on the isomorphism class of $s$, at least up to homotopy.

**Example 5.2.2.** To illustrate the second defect, let $X$ be a non-empty anima, and fix $x_0 \in X_0$ an object. Then the functor

$$f : X \to \mathrm{pt} \xrightarrow{x_0} X$$

has strict fiber $X$ above $x_0$, and has strict fiber $\emptyset$ above every other object. In particular, if $x_0'$ is any object of $X$ which is isomorphic but not equal to $x_0$, then the strict fibers of $f$ above $x_0$ and $x_0'$ are completely different.

Both of these defects are rather serious. Let's focus on the second one for now. To get an isomorphism-invariant notion of fiber, we might define the fiber of $f : X \to S$ above $s \in S_0$ to be something like the sub-simplicial set of $X$ spanned by the vertices which map to something *isomorphic* to $s$. However, it is contrary to the philosophy of anima to say "there exists an isomorphism" – the whole point of using $\infty$-groupoids was to keep track of all the reasons two things can be equivalent.

Thus, the natural notion of fiber for a functor between $\infty$-groupoids is to encode a pair of a point $x$ in $X$ *plus an isomorphism* from $f(x)$ to $s$. Thus, we need a simplicial set encoding maps $\Delta_1 \to S$ with a *fixed endpoint*. For this, we introduce *pointed mapping spaces*.

### 5.2.2 Digression: pointed mapping spaces

**Definition 5.2.3.** A *pointed simplicial set* is a simplicial set $Y$ together with a distinguished 0-simplex $y_0$.

**Definition 5.2.4.** If $(X, x), (Y, y)$ are pointed simplicial sets, we define the *pointed mapping space* $[(X, x), (Y, y)]_*$ to be the fiber of

$$\mathrm{eval}_x : [X, Y] \to Y$$

above $y$.

We view $[(X, x), (Y, y)]_*$ as being pointed, with the basepoint being the constant map with value $y$.

**Remark 5.2.5.** By Theorem 4.8.13, if $Y$ is a Kan complex, then so is $[(X, x), (Y, y)]_*$.

**Definition 5.2.6.** We say two maps $f, g : (X, x) \to (Y, y)$ are *pointed homotopic* if they lie in the same component of $[(X, x), (Y, y)]_*$.

While we're here, we explain a pointed version of the tensor-hom adjunction; we don't need it to understand homotopy fibers, but it is good to keep in mind since the tensor-hom adjunction is such a basic tool.

**Ideation 5.2.7.** There are two natural products of pointed spaces: we could declare

$$(X, x) \times (Y, y) = (X \times Y, (x, y)),$$

for example. However, maps $(X, x) \times (Y, y) \to (Z, z)$ do *not* correspond to maps $(X, x) \to [(Y, y), (Z, z)]_*$.

The trouble is that any map $(X, x) \to [(Y, y), (Z, z)]_*$ must send $x$ to the constant map $(Y, y) \to (Z, z)$. But a pointed map $(X \times Y, (x, y)) \to (Z, z)$ need only send $(x, y)$ to $z$; it doesn't need to send any other point $(x, y')$ to $z$.

To fix this, we introduce the *smash product*.

**Definition 5.2.8.** The *smash product* of pointed simplicial sets $(X, x), (Y, y)$ is the pointed simplicial set

$$(X, x) \wedge (Y, y) := (X \times Y) \coprod_{(X \times \{y\}) \cup (\{x\} \times Y)} \mathrm{pt} = \frac{X \times Y}{(X \times \{y\}) \cup (\{x\} \times Y)},$$

with pt being the base point.

**Remark 5.2.9.** The smash product collapses $X \times \{y\}$ and $\{x\} \times Y$ down to a common point.

**Theorem 5.2.10** (Pointed tensor-hom adjunction)**.** *Take* $(X, x), (Y, y), (Z, z)$ *three pointed simplicial sets. Then*

$$[X \wedge Y, Z]_* = [X, [Y, Z]_*]_*.$$

*Proof.* An $n$-simplex of $[X \wedge Y, Z]_*$ is a map

$$\varphi : \Delta_n \times (X \wedge Y) \to Z$$

such that $\varphi|_{\Delta_n \times \mathrm{pt}}$ is constant with value $z$, for pt the basepoint of $X \wedge Y$.

By definition of the smash product, this $\varphi$ is further equivalent to a map

$$\varphi' : \Delta_n \times X \times Y \to Z$$

such that $\varphi'|_{\Delta_n \times X \times \{y\}}$ and $\varphi'|_{\Delta_n \times \{x\} \times Y}$ are again both constant with value $z$.

Similarly, an $n$-simplex of $[X, [Y, Z]_*]_*$ is just a map

$$\varphi : \Delta_n \times X \times Y \to Z$$

obeying the same restriction, and so we get an isomorphism of simplicial sets. □

### 5.2.3 Homotopy fibers

With pointed mapping spaces, we can finally make formal our description of the homotopy fiber.

**Definition 5.2.11.** Fix $(S, s)$ a pointed simplicial set. The *homotopy fiber* of a morphism $f : X \to S$ above $s$ is the simplicial set

$$F := X \times_S [(\Delta_1, 1), (S, s)]_*.$$

Here, the map $[\Delta_1, S] \to S$ is evaluation at 0.

**Remark 5.2.12.** When $X$ has a basepoint $x$ satisfying $f(x) = s$, we can view $F$ as pointed via the base point $(x, c_{fx})$, where $c_{fx} \in [\Delta_1, S]_*$ is the degenerate edge of $S$ associated to the vertex $fx$.

**Warning 5.2.13.** We often abuse notation and write just $x$ for the basepoint of $F$.

**Example 5.2.14** (Comparison of notions of fiber)**.** Let $\mathcal{G}$ denote the 1-groupoid consisting of one object having automorphism group $\mathbb{Z}/2$. Let pt denote the 1-groupoid with one object having automorphism group 1.

The homotopy fiber of the unique functor $f : \mathrm{pt} \to \mathcal{G}$ is the groupoid $\{0, 1\}$ consisting of two non-isomorphic objects, neither of which has a non-identity automorphism. If we used the strict fiber, or if we used the 'sub-simplicial set of things isomorphic to...' definition briefly considered above, we'd get pt instead.

Why is $\{0, 1\}$ a better fiber than pt, though?

Well, whether or not you think $\{0, 1\}$ is a better fiber for $\mathrm{pt} \to \mathcal{G}$ depends on what you want your fibers to do. In set theory, one common way to use fibers is to check if a function is a bijection. Here, we have a functor $f$ whose strict fiber is a point; as there's only one fiber, we might expect $f$ to be an equivalence, from our set theory intuition... but of course this is false, $f$ is not an equivalence of groupoids! The homotopy fiber shows us the obstruction to $f$ being a homotopy equivalence: $\mathcal{G}$ has an extra isomorphism that pt does not have, encoded as the extra point of our homotopy fiber.

**Example 5.2.15.** For a group $G$, write $BG$ for the 1-groupoid having one object $*$ of automorphism group $G$. Any group homomorphism $\varphi : G \to G'$ gives rise to a functor

$$f : BG \to BG'.$$

Indeed, the homotopy fiber has as its objects pairs $(*, g')$ where $g' \in G'$. An isomorphism $(*, g'_0) \simeq (*, g'_1)$ is the data of a $g \in G$ such that $g'_1 \varphi(g) = g'_0$.

Thus $(*, g'_0)$ and $(*, g'_1)$ are isomorphic if and only if they lie in the same left coset of $\operatorname{im} \varphi$, meaning the homotopy fiber of $f$ has set of isomorphism classes $G'/\operatorname{im} \varphi$ (here we mean the set of left cosets; this image is of course not always normal, and so this set does not always have a group structure).

Moreover, the automorphism group of $(*, g')$ is just $\ker(\varphi)$. So, the $\pi_0$ of our homotopy fiber measures the failure of $\varphi$ to be surjective, and the stabilizers of our homotopy fiber measure the failure of $\varphi$ to be injective.

The homotopy fiber of a functor between $\infty$-groupoids is always an $\infty$-groupoid.

**Proposition 5.2.16.** *Let $f : X \to S$ be a map of Kan complexes, and let $s \in S_0$ be a basepoint. Then its homotopy fiber $F = X \times_S [(\Delta_1, 1), (S, s)]_*$ is a Kan complex.*

*Proof.* The map

$$(\mathrm{eval}_0, \mathrm{eval}_1) : [\Delta_1, S] \to S \times S$$

is a Kan fibration by Theorem 4.8.13, and so its pullback along $X \times \{s\} \hookrightarrow X \times S \xrightarrow{f \times \mathrm{id}_S} S \times S$ is also a Kan fibration. But this pullback is just $F$, and so $F \to X$ is a Kan fibration; as Kan fibrations are stable under composition, $F$ is thus a Kan complex. □

## 5.3 Fibers of Kan fibrations

Recall the strict fiber has two disadvantages to the homotopy fiber: it is not always an $\infty$-groupoid, and it is not always an isomorphism invariant of the point you were taking the fiber at.

When $f : X \to S$ is a *Kan fibration*, though, all of its strict fibers are $\infty$-groupoids. Thus the first of the defects of the strict fiber is fixed in this case. Remarkably, the second defect is also fixed in the case of Kan fibrations!

**Construction 5.3.1.** Let $f : X \to S$ be a Kan fibration of *Kan complexes*, and assume $p : \Delta_1 \to S$ is a path from $s$ to $s'$.

We construct a *homotopy class* of maps $X_s \to X_{s'}$. We denote this homotopy class by $\psi_p \in \pi_0[X_s, X_{s'}]$.

Consider the commutative diagram

$$\begin{array}{ccc} X_s \times 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ X_s \times \Delta_1 & \longrightarrow & S, \end{array}$$

where the top horizontal map is inclusion and the bottom horizontal map is adjunct to the constant map $X_s \to [\Delta_1, S]$ with value $p$.

By Lemma 4.7.12, the map $X_s \times \{0\} \hookrightarrow X_s \times \Delta_1$ is anodyne. Thus we may find a map

$$\varphi : X_s \times \Delta_1 \to X$$

which has the following two properties:

1. $\varphi|_{X_s \times \{0\}}$ is just the inclusion $X_s \hookrightarrow X$,
2. the composite $f \circ \varphi$ is just the composition

$$X_s \times \Delta_1 \to \Delta_1 \xrightarrow{p} S.$$

In particular, $\varphi|_{X_s \times \{1\}} : X_s \to X$ factors through $X_{s'}$, giving us a map

$$\psi_p : X_s \to X_{s'}.$$

Theorem 4.8.9 implies that $\psi_p$ is well-defined up to homotopy, as desired.

**Theorem 5.3.2.** *Let $f : X \to S$ be a Kan fibration of* Kan complexes. *Suppose that $p, q : \Delta_1 \to S$ are paths in $S$, with $p : s \to s'$ and $q : s' \to s''$.*

*Then, in the notation of Construction 5.3.1,*

1. *if $r : s \to s''$ is any composition (see Definition 4.4.4) of $q$ and $p$,*

$$\psi_r \simeq \psi_q \circ \psi_p,$$

2. *if $\mathrm{id}_s : s \to s$ is the constant path, then $\psi_{\mathrm{id}_s} \simeq \mathrm{id}_{X_s}$.*

*In particular, each $\psi_p$ is a homotopy equivalence, with homotopy inverse $\psi_{p^{-1}}$.*

**Remark 5.3.3.** Composition of homotopy classes of maps makes sense by Proposition 4.9.3.

*Proof.* The second fact is easy: Construction 5.3.1 constructs $\psi_{\mathrm{id}_s}$ as the solution to a certain lifting problem, but when our path is degenerate, the map

$$\varphi : X_s \times \Delta_1 \to X_s \hookrightarrow X$$

(with first arrow projection) solves our lifting problem.

For the first fact, take

$$\varphi_p : X_s \times \Delta_1 \to X,$$
$$\varphi_q : X_{s'} \times \Delta_1 \to X$$

solutions to the lifting problems used to define $\psi_p, \psi_q$.

Choose a 2-simplex $\sigma : \Delta_2 \to S$ witnessing $r \simeq q \circ p$.

We define

$$\alpha : X_s \times \Lambda^1_2 \to X$$

as follows. On the restriction to the 01 edge of $\Lambda^1_2$, let $\alpha$ be $\varphi_p$. On the restriction to the 12 edge, let $\alpha$ be the composite map

$$X_s \times \Delta_1 \xrightarrow{\psi_p \times \mathrm{id}_{\Delta_1}} X_{s'} \times \Delta_1 \xrightarrow{\varphi_q} X.$$

These two maps glue because on the vertex 1, both are $\psi_p$.

As $\Lambda^1_2 \hookrightarrow \Delta_2$ is anodyne, by Lemma 4.7.12 we may extend $\alpha$ to a map

$$\alpha' : X_s \times \Delta_2 \to X$$

such that

$$f \circ \alpha' : X_s \times \Delta_2 \to S$$

is the map $X_s \times \Delta_2 \to \Delta_2 \xrightarrow{\sigma} S$.

The 02 edge of this map $\alpha'$ solves the lifting problem used to construct $\psi_r$, and in particular $\psi_r \simeq \alpha'|_{X_s \times \{2\}}$.

The restriction of $\alpha'$ to $X_s \times \{2\}$ is given by $\psi_q \circ \psi_p$, though. Thus $\psi_r \simeq \psi_q \circ \psi_p$, as desired. □

Theorem 5.3.2 allows us to state the following result, showing that homotopic morphisms of Kan fibrations induce homotopic morphisms of fibers.

**Theorem 5.3.4.** *Suppose we have a commutative diagram*

$$\begin{array}{ccc} X \times \Delta_1 & \xrightarrow{H_f} & X' \\ \downarrow{\scriptstyle \theta \times \mathrm{id}_{\Delta_1}} & & \downarrow{\scriptstyle \theta'} \\ Y \times \Delta_1 & \xrightarrow{H_g} & Y', \end{array}$$

*where $\theta : X \to Y$ and $\theta' : X' \to Y'$ are both Kan fibrations and $Y, Y'$ are anima. Let $f_0, f_1 : X \to X'$ and $g_0, g_1 : Y \to Y'$ be the endpoints of $H_f, H_g$.*

*Take $y \in Y$. Set $y'_0 = g_0(y), y'_1 = g_1(y)$. The homotopy $H_g$ gives a path $p : y'_0 \to y'_1$ in $Y'$. Let*

$$\psi_p : (\theta')^{-1}(y'_0) \to (\theta')^{-1}(y'_1)$$

*be any representative of the homotopy class of maps constructed in Construction 5.3.1.*

*Then, the diagram*

$$\begin{array}{ccc} \theta^{-1}(y) & \xrightarrow{f_0} & (\theta')^{-1}(y'_0) \\ & \searrow^{f_1} & \downarrow{\scriptstyle \psi_p} \\ & & (\theta')^{-1}(y'_1) \end{array}$$

*commutes up to homotopy; that is,*

$$f_1 \simeq \psi_p \circ f_0.$$

*Proof.* To construct $\psi_p$, we start by finding a map

$$\varphi : (\theta')^{-1}(y'_0) \times \Delta_1 \to X'$$

lying above $p$ and which, at time zero, is inclusion. Then $\psi_p$ is $\varphi$ restricted to time 1. Thus $\psi_p \circ f_0$ is obtained by taking

$$\varphi \circ (f_0 \times \mathrm{id}_{\Delta_1}) : \theta^{-1}(y) \times \Delta_1 \to X'$$

and restricting to time 1.

Observe that $\varphi \circ f_0$ is a map to $X'$ whose 0 endpoint is just the composite map $\theta^{-1}(y) \xrightarrow{f_0} (\theta')^{-1}(y'_0) \hookrightarrow X'$, and whose projection to $Y'$ is $p$.

But $H_f$ restricts to a map

$$H_f : \theta^{-1}(y) \times \Delta_1 \to X'$$

having those same properties: its zero endpoint is just $f_0$, and the commutativity of our diagram tells us that $\theta' \circ H_f = H_g \circ \theta$ which, on $\theta^{-1}(y)$, is just $p$.

Thus $H_f, \varphi \circ f_0$ both solve the same lifting problem of an anodyne map against a Kan fibration $X' \to Y'$. By Theorem 4.8.9 (applied to the strongly anodyne map $\{0\} \hookrightarrow \Delta_1$ and the Kan fibration $[\theta^{-1}(y), X'] \to [\theta^{-1}(y), Y']$), these maps are therefore homotopic; restricting any homotopy $H_f \simeq \varphi \circ f_0$ to $(\theta^{-1}(y) \times \{1\}) \times \Delta_1$ then gives us our desired homotopy $f_1 \simeq \psi_p \circ f_0$. □

**Corollary 5.3.5.** *Assume*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} \\ Y & \xrightarrow{g} & Y' \end{array}$$

*is a commutative square in which $\theta, \theta'$ are both Kan fibrations of anima, and $f, g$ are both homotopy equivalences. Assume* also *that we can choose homotopy inverses $f', g'$ of $f, g$ obeying the following compatibilities:*

1. *the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ Y' & \xrightarrow{g'} & Y \end{array}$$

   *commutes,*

2. *we can find homotopies $H' : g'g \simeq \mathrm{id}_Y, H : f'f \simeq \mathrm{id}_X$ such that the diagram*

$$\begin{array}{ccc} X \times \Delta_1 & \xrightarrow{H} & X \\ \downarrow{\scriptstyle \theta \times \mathrm{id}_{\Delta_1}} & & \downarrow{\scriptstyle \theta} \\ Y \times \Delta_1 & \xrightarrow{H'} & Y \end{array}$$

   *commutes,*

3. *we can find homotopies $G' : gg' \simeq \mathrm{id}_{Y'}, G : ff' \simeq \mathrm{id}_{X'}$ such that the diagram*

$$\begin{array}{ccc} X' \times \Delta_1 & \xrightarrow{G} & X' \\ \downarrow{\scriptstyle \theta' \times \mathrm{id}_{\Delta_1}} & & \downarrow{\scriptstyle \theta'} \\ Y' \times \Delta_1 & \xrightarrow{G'} & Y' \end{array}$$

   *commutes.*

*Then, for any $y \in Y$, the map*

$$f_y : \theta^{-1}(y) \xrightarrow{f} (\theta')^{-1}(gy)$$

*is also a homotopy equivalence.*

**Remark 5.3.6.** Of course, in practice one wants to remove the assumption that the homotopy inverses $f', g'$ can be chosen in a compatible way, as that is annoying to check; in Theorem 9.5.3, we will prove a variant of this corollary with no compatibility assumed.

*Proof.* Consider the map

$$f'_{gy} : (\theta')^{-1}(gy) \to \theta^{-1}(g'gy).$$

The homotopy $H' : g'g \simeq \mathrm{id}_Y$ gives us a path $p : g'gy \to y$. By Theorem 5.3.4, we have

$$\psi_p \circ (f'f)_y \simeq \mathrm{id}_{\theta^{-1}(y)}.$$

Thus

$$\mathrm{id} \simeq \psi_p \circ (f'f)_y = (\psi_p \circ f'_{gy}) \circ f_y,$$

so that $\psi_p \circ f'_{gy}$ is always the left homotopy inverse to $f_y$.

Set $y' = gy$. The homotopy $G'$ gives us a path $q : gg'y' \to y'$. Again by Theorem 5.3.4, we have

$$\mathrm{id} \simeq \psi_q \circ (ff')_{y'} = \psi_q \circ f_{g'y'} \circ f'_{y'},$$

implying that $f'_{y'} = f'_{gy}$ admits a left homotopy inverse.

But $\psi_p$ is a homotopy equivalence; hence $\psi_p \circ f'_{gy}$ admits both a right homotopy inverse ($f_y$) and a left one, making it a homotopy equivalence. But then $f_y$ is the right homotopy inverse to a homotopy equivalence, forcing $f_y$ to itself be a homotopy equivalence, as desired. □

### 5.3.1 Locally trivial fibrations

Theorem 5.3.2 is why Kan fibrations are called *fibrations*. More precisely, recall the usual topological notion of a locally trivial fibration.

**Definition 5.3.7.** A continuous function $f : X \to S$ of point-set topological spaces is said to be a *locally trivial fibration*, with *fiber* $F$ (for $F$ a point-set topological space) if for every $s \in S$, there is some open $U \ni s$ such that $f^{-1}(U)$ is homeomorphic to $U \times F$ by a homeomorphism carrying $f$ to the projection function $U \times F \to U$.

A locally trivial fibration is a continuous map all of whose fibers are *isomorphic* (in a very strong way). Similarly, a Kan fibration is a morphism of simplicial sets whose fibers are all *homotopic* in a very strong way, thanks to Theorem 5.3.2. Thus, a Kan fibration is just a "homotopy fibration."

To make this a bit more precise, in the rest of this section we prove that Sing turns the locally trivial fibrations of topology into Kan fibrations.

**Lemma 5.3.8.** *Every locally trivial fibration over $K = [0,1]^d$ is trivial.*

*Proof.* Let $p : E \to K$ be a fibration, with fiber $F$. By the Lebesgue number lemma, there is some $r > 0$ so that $p$ is trivial on every subset of diameter at most $2r$.

We first prove this when $K = [0, 1]$. In that case, cover $K$ by the closed sets $[0, r], [r, 2r], ....$

Choose a trivialization $h_0 : p^{-1}([0, r]) \simeq [0, r] \times F$ and a trivialization $h_1 : p^{-1}([r, 2r]) \simeq [r, 2r] \times F$. Define

$$h : [0, 2r] \times F \simeq p^{-1}([0, 2r])$$

as follows. Let

$$\varphi : F \to F$$

denote the map

$$\varphi(f) := p_F(h_1(h_0^{-1}(r, f))).$$

Note that $\varphi$ is a homeomorphism (with inverse $f \mapsto p_F(h_0(h_1^{-1}(r, f)))$).

Define

$$h(t, f) := \begin{cases} h_0^{-1}(t, f) & t \leq r, \\ h_1^{-1}(t, \varphi(f)) & t \geq r. \end{cases}$$

When $t = r$, the two formulas coincide (which is why $h$ is continuous!), as

$$h_1^{-1}(r, \varphi(f)) = h_1^{-1}(r, p_F(h_1(h_0^{-1}(r, f)))).$$

As $h_0, h_1$ are both compatible with the projection to the interval, we deduce that $h_1(h_0^{-1}(r, f))$ has time component $r$, so

$$h_1^{-1}(r, p_F(h_1(h_0^{-1}(r, f)))) = h_0^{-1}(r, f),$$

as desired. Combining this patching lemma repeatedly, our bundle is trivial over $[0, 1]$.

For higher dimensions, tile $K$ by small $d$-cubes, and find some sequence of adjacent cubes, then repeatedly apply the above strategy to finish; a small patching argument is needed, which we leave to the reader (induct on $d$). □

**Theorem 5.3.9** (Fiber bundles give Kan fibrations)**.** *Let $f : X \to S$ be a locally trivial fibration with fiber $F$. Then $\mathrm{Sing}(f) : \mathrm{Sing}(X) \to \mathrm{Sing}(S)$ is a Kan fibration.*

*Proof.* It suffices to prove that every lifting problem

$$\begin{array}{ccc} |\Lambda^i_n| & \xrightarrow{\tau'} & X \\ \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow{\sigma} & S \end{array}$$

can be solved. But in fact, it suffices to solve lifting problems against $0 \times \Delta^n \hookrightarrow [0, 1] \times \Delta^n$. Indeed, $|\Lambda^i_n| \hookrightarrow \Delta^n$ is a retract of $\{0\} \times \Delta^n \hookrightarrow [0, 1] \times \Delta^n$, via the continuous map

$$r_i : [0, 1] \times \Delta^n \to \Delta^n,$$

$$r_i(t, x_0, ..., x_n) = (x_0 - \lambda, ..., x_{i-1} - \lambda, x_i + n\lambda, x_{i+1} - \lambda, ..., x_n - \lambda),$$

where

$$\lambda = \max\{0, \min\{x_0, ..., x_{i-1}, x_{i+1}, ..., x_n\} - t\}.$$

As $|\Lambda^i_n|$ is the subset of $\Delta^n$ consisting of points having some $x_j = 0$ with $j \neq i$, the map $r_i$ sends $\{0\} \times \Delta^n$ onto $|\Lambda^i_n|$. It is easy to see it is a retraction, and so we conclude.

So, it suffices to prove every lifting problem of the form

$$\begin{array}{ccc} \{0\} \times \Delta^n & \xrightarrow{\tau'} & X \\ \downarrow & & \downarrow f \\ \times \Delta^n & \xrightarrow{\sigma} & S \end{array}$$

can be solved. Define $\sigma^* X := X \times_S ([0,1] \times \Delta^n)$. Note that $\tau'$ gives a continuous map

$$\tilde{\tau} : \{0\} \times \Delta^n \to \sigma^* X.$$

The pullback of a fiber bundle is clearly a fiber bundle, so $\sigma^* X \to [0,1] \times \Delta^n$ is a fiber bundle; but the base is homeomorphic to $[0,1]^{n+1}$, and so by Lemma 5.3.8 this is a trivial bundle. Thus we can find a homeomorphism

$$h : \sigma^* X \simeq [0,1] \times \Delta^n \times F.$$

Define

$$s : [0,1] \times \Delta^n \to [0,1] \times \Delta^n \times F$$

as

$$s(t,x) = (t, x, p_F h \tilde{\tau}(0,x)).$$

Composing $s$ with projection to $X$ and $h^{-1}$ solves our lifting problem, so we conclude. □

## 5.4 Homotopy fibers as strict fibers

For Kan fibrations, the strict fiber is homotopically well-behaved. This suggests that the strict fiber should be equivalent to the homotopy fiber in the case of Kan fibrations, and indeed this is the case.

**Theorem 5.4.1.** *Let $f : X \to S$ be a* Kan fibration *of anima. Choose $s \in S_0$. Then the natural inclusion*

$$i : X_s \hookrightarrow F$$

*from the strict fiber above $s$ into the homotopy fiber above $s$ is a homotopy equivalence of anima.*

**Remark 5.4.2.** A map $X_s \to F$ is determined by its projections to $X$ and $[(\Delta_1, 1), (S, s)]_*$; the inclusion $i$ is the map whose projection to $X$ is $X_s \hookrightarrow X$, and whose projection to $[(\Delta_1, 1), (S, s)]_*$ is the constant map with value $c_s$, for $c_s$ the constant edge at $s$.

*Proof.* We produce a homotopy inverse

$$r : F \to X_s$$

of $i$.

A point of $F$ is a pair $(x, p)$, where $x$ is some point of $X$, and $p$ is some path from $f(x)$ to $s$. Intuitively, our map $r$ will lift $p$ to a path in $X$, and then send $(x, p)$ to the right endpoint of that lift.

Recall that

$$F = X \times_S [(\Delta_1, 1), (S, s)]_*.$$

As $\{0\} \hookrightarrow \Delta_1$ is strongly anodyne, Theorem 4.8.8 tells us that $(F \times \{0\}) \cup (X_s \times \Delta_1) \hookrightarrow F \times \Delta_1$ is anodyne as well. Thus we can solve the lifting problem

$$\begin{array}{ccc} (F \times \{0\}) \cup (X_s \times \Delta_1) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ F \times \Delta_1 & \longrightarrow & S, \end{array}$$

where

1. the top horizontal map $F \to X$ is just projection onto $X$,
2. the top horizontal map $X_s \times \Delta_1 \to X$ is just the adjunct to the map $X_s \to [\Delta_1, X]$ sending every point to the associated constant map,
3. the bottom map $F \times \Delta_1 \to S$ is adjunct to the composite

$$F \to [\Delta_1, S]_* \hookrightarrow [\Delta_1, S].$$

Let

$$\tilde{r} : F \times \Delta_1 \to X$$

be such a solution. Define $r' = \tilde{r}|_{F \times \{1\}}$.

Observe that $r' : F \to X$ factors through $j : X_s \hookrightarrow X$, by compatibility of $\tilde{r}$ with projection to $S$. Thus we can write $r' = j \circ r$ for some unique map

$$r : F \to X_s.$$

Then $r \circ i = \mathrm{id}_{X_s}$ (strictly!). We now construct a homotopy $ir \simeq \mathrm{id}_F$. First, let

$$m : \Delta_1 \times \Delta_1 \to \Delta_1$$

be given by the monotone map $m(t, u) = \max(t, u)$.

Define

$$H : F \times \Delta_1 \to F$$

the homotopy

$$H((x, p), t) := (\tilde{r}((x, p), t), (t' \mapsto p(m(t, t')))).$$

Intuitively, this $H$ forces $p$ to start at time $t$.

Note that $t' \mapsto p(m(t, t'))$ is a path in $S$ from $p(t)$ to $p(1)$, and $\tilde{r}((x, p), t)$ is a point of $X$ lying above $p(t)$, which is why our formula is valued in $F$. At time 0, $H$ is just $\mathrm{id}_F$, and at time 1, $H$ is $ir$, so we have our desired homotopy. □

**Corollary 5.4.3.** *Let $f : X \to S$ be a Kan fibration of anima whose homotopy fiber $F$ above some $s \in S$ is non-empty. Then the strict fiber $X_s$ is non-empty.*

*In fact, if $(x, p : f(x) \to s)$ is any point of the homotopy fiber, then we may find some path $q : x \to x'$ in $X$ where $x' \in X_s$ and $f(q) = p$.*

**Remark 5.4.4.** This is more of a porism than a corollary.

*Proof.* Observe that $i : X_s \hookrightarrow F$ is a homotopy equivalence, with a homotopy inverse $r : F \to X_s$ constructed above. This homotopy inverse was constructed by lifting $p$ to a path $q : x \to x'$ where $x' \in X_s$; as $q$ lifts $p$, we have $f(q) = p$. □

**Corollary 5.4.5.** *Let $f : X \to S$ be a Kan fibration between Kan complexes, and $i : A \hookrightarrow B$ an inclusion. Suppose we have a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\bar{b}} & S. \end{array}$$

*Suppose that this lifting problem has a solution* up to homotopy*: that is, suppose we have a map $b' : B \to X$ and homotopies $H_1 : f \circ b' \simeq \bar{b}, H_2 : b'|_A \simeq a$, such that the restriction of $H_1$ to $A$ is equal to $f \circ H_2$.*

*Then we may find a map $b : B \to X$ such that $b$ is a literal solution to this lifting problem, and so that $b \simeq b'$. In fact, we can even choose a homotopy*

$$H : \Delta_1 \times B \to X$$

*from $b'$ to $b$ such that $f \circ H$ is the homotopy $H_1$, and such that $H|_{\Delta_1 \times A}$ is $H_2$.*

*Proof.* By Theorem 4.8.13,

$$\theta : [B, X] \to [A, X] \times_{[A,S]} [B, S]$$

is a Kan fibration.

Observe that $(b', (H_2, H_1))$ is a point of the homotopy fiber of $\theta$ above $(a, \bar{b})$. By Corollary 5.4.3, it follows that the strict fiber of $\theta$ above $(a, \bar{b})$ is non-empty, and even more strongly we may find a point $b \in [B, X]_0$ of the strict fiber together with a path

$$H : b' \to b$$

having $\theta(H) = (H_2, H_1)$.

Unwinding definitions, this $H$ is just a homotopy from $b'$ to $b$. To say that $\theta(H) = (H_2, H_1)$ is to say the restriction of this homotopy to $A$ is $H_2$ and the projection of it to $S$ is $H_1$, so we conclude. □

## 5.5 Fibrant replacement

Let $f : X \to S$ be a morphism of anima, and fix $s \in S_0$. The strict fiber $X_s$ is easy to compute, but poorly behaved. The homotopy fiber $X \times_S [(\Delta_1, 1), (S, s)]_*$ is very well-behaved, but annoying to work with explicitly, or even to prove theorems about. Theorem 5.4.1 tells us that, when $f$ is a Kan fibration, we have the best of both worlds: the homotopy fiber and the strict fiber are homotopy equivalent, so you can use the strict fiber $X_s$ but enjoy all the benefits of the homotopy fiber.

This is such a useful trick that we now give a general machine for replacing functors between anima by an equivalent *Kan fibration.*

**Theorem 5.5.1** (Fibrant replacement for anima)**.** *Let $f : X \to S$ be a morphism of anima. Define*

$$Pf := X \times_S ([\Delta_1, S], \mathrm{eval}_0).$$

*Then*

1. *the map $f' : Pf \to [\Delta_1, S] \xrightarrow{\mathrm{eval}_1} S$ is a Kan fibration (and in particular, $Pf$ is an anima),*

2. *the inclusion $i : X \hookrightarrow Pf$ (use the constant path at $f(x)$ for the second factor) is a homotopy equivalence,*

3. *and the homotopy fiber of $f$ at any point $s \in S$ is equal to the strict fiber of $f'$ at $s$.*

*Proof.* The base change of

$$[\Delta_1, S] \to S \times S$$

along $X \times S \xrightarrow{f \times \mathrm{id}_S} S \times S$ is just

$$Pf \to X \times S.$$

Thus $Pf \to X \times S$ is a Kan fibration; as $X$ is a Kan complex, $X \times S \to S$ is a Kan fibration as well, and so $f'$ is a Kan fibration.

For the second point, the projection

$$r : Pf \to X$$

is actually a retraction of $i$. Indeed, $ri = \mathrm{id}_X$ strictly. To construct a homotopy $ir \simeq \mathrm{id}_{Pf}$, we use a similar trick to the proof of Theorem 5.4.1. Let

$$H : Pf \times \Delta_1 \to Pf$$

be

$$H((x, p), t) = (x, t' \mapsto p(\min(t, t'))).$$

This is our desired homotopy.

The third point is immediate from unwinding definitions. □

**Remark 5.5.2.** In certain theoretical arguments, it will be very helpful to replace $X$ by $Pf$ to reduce to the case of Kan fibrations.

Using fibrant replacement, we can give very quick proofs of homotopy fiber versions of all the results of Section 5.3.

**Construction 5.5.3.** Let $f : X \to S$ be a functor between anima. Suppose $p : s \to s'$ is a path in $S$. Let $F, F'$ denote the homotopy fibers of $f$ above $s, s'$.

We define a homotopy class of maps $\psi_p : F \to F'$ as follows. By fibrant replacement, $f' : Pf \to S$ is a Kan fibration; so Construction 5.3.1 gives us a homotopy class of maps $\psi_p : (f')^{-1}(s) \to (f')^{-1}(s')$. Using the identification of strict fibers of $f'$ with homotopy fibers of $f$, we conclude.

**Theorem 5.5.4.** *Let $f : X \to S$ be a functor between anima. Suppose that $p, q : \Delta_1 \to S$ are paths in $S$, with $p : s \to s'$ and $q : s' \to s''$.*

*Then, in the notation of Construction 5.5.3,*

1. *if $r : s \to s''$ is any composition (see Definition 4.4.4) of $q$ and $p$,*

$$\psi_r \simeq \psi_q \circ \psi_p,$$

2. *if $\mathrm{id}_s : s \to s$ is the constant path, then $\psi_{\mathrm{id}_s} \simeq \mathrm{id}_F$.*

*In particular, each $\psi_p$ is a homotopy equivalence, with homotopy inverse $\psi_{p^{-1}}$.*

*Proof.* Apply Theorem 5.3.2 to the fibrant replacement of $f$. □

**Theorem 5.5.5.** *Suppose we have a commutative diagram*

$$\begin{array}{ccc} X \times \Delta_1 & \xrightarrow{H_f} & X' \\ \downarrow{\scriptstyle \theta \times \mathrm{id}_{\Delta_1}} & & \downarrow{\scriptstyle \theta'} \\ Y \times \Delta_1 & \xrightarrow{H_g} & Y', \end{array}$$

*where $X, X', Y, Y'$ are anima. Let $f_0, f_1 : X \to X'$ and $g_0, g_1 : Y \to Y'$ be the endpoints of $H_f, H_g$.*

*Take $y \in Y$. Set $y'_0 = g_0(y), y'_1 = g_1(y)$. The homotopy $H_g$ gives a path $p : y'_0 \to y'_1$ in $Y'$. Put $F'_0, F'_1$ the homotopy fibers of $\theta'$ above $y'_0$ and $y'_1$. Let*

$$\psi_p : F'_0 \to F'_1$$

*be any representative of the homotopy class of maps constructed in Construction 5.5.3.*

*Take $F$ the homotopy fiber of $\theta$ above $y$. Then, the diagram*

$$\begin{array}{ccc} F & \xrightarrow{(f_0, g_0)} & F'_0 \\ & \searrow^{(f_1, g_1)} & \downarrow{\scriptstyle \psi_p} \\ & & F'_1 \end{array}$$

*commutes up to homotopy; that is,*

$$(f_1, g_1) \simeq \psi_p \circ (f_0, g_0).$$

*Proof.* Apply Theorem 5.3.4 to the fibrant replacements. □

# Chapter 6

# Morphisms and composition in an $\infty$-groupoid

## 6.1 Introduction

In a groupoid, between any two objects, we have a set of morphisms. And in a 2-groupoid, between any two objects, we have a groupoid of morphisms.

It stands to reason that, in an $\infty$-groupoid, between any two objects there is some $\infty$-groupoid of morphisms. In this chapter, we construct this $\infty$-groupoid of morphisms, and try to get a concrete handle on it.

## 6.2 Morphism spaces

The last chapter studied fibers of functors between $\infty$-groupoids. As our first application of this study, we define morphisms as follows.

**Definition 6.2.1.** Let $X$ be an $\infty$-groupoid, and $x, x' \in X_0$. The *morphism space* $\mathrm{Mor}_X(x, x')$ is the strict fiber of the Kan fibration

$$(\mathrm{eval}_0, \mathrm{eval}_1) : [\Delta_1, X] \to X \times X$$

above $(x, x')$.

**Notation 6.2.2.** When $x = x'$, we refer to $\mathrm{Mor}_X(x, x)$ as the *loop space*, and often denote it by $\Omega(X, x)$, or even just $\Omega X$ when $x$ is understood.

**Notation 6.2.3.** When thinking more geometrically, we will write $P_{x,x'}$ for $\mathrm{Mor}_X(x, x')$, and refer to it as the *path space*.

While these morphism spaces can be quite different from $X$, the space $[\Delta_1, X]$ of all arrows is actually equivalent to $X$.

**Proposition 6.2.4.** *Let $X$ be an $\infty$-groupoid. Then $X$ and $[\Delta_1, X]$ are homotopy equivalent; in fact, the evaluation at 0 functor*

$$[\Delta_1, X] \to X$$

*is a homotopy equivalence.*

*Proof.* By Theorem 4.8.13, it is a trivial Kan fibration. □

In fact, Proposition 6.2.4 holds with $\Delta_1$ replaced by an arbitrary simplex.

**Lemma 6.2.5.** *If $X$ is an anima, then the evaluation at 0 functor*

$$\theta : [\Delta_n, X] \to X$$

*is a homotopy equivalence.*

*Proof.* Let $i : 0 \hookrightarrow \Delta_n$ be the inclusion, and $r : \Delta_n \to \mathrm{pt}$ be the map projecting to a point. Clearly $r \circ i = \mathrm{id}_{\mathrm{pt}}$. Moreover, $ir$ is naively homotopic to $\mathrm{id}$. Indeed, let

$$H : \Delta_1 \times \Delta_n \to \Delta_n$$

given as the monotone map

$$H(i,j) = \begin{cases} 0 & i = 0, \\ j & i = 1. \end{cases}$$

Then $H$ is a homotopy from $ir$ to id, as desired.

Define

$$r^* : X \to [\Delta_n, X]$$

the map obtained by precomposition with $r$. Then $ri = \mathrm{id}$ implies $i^* r^* = \mathrm{id}$. The homotopy $H$ can be used to construct a homotopy

$$H^* : \Delta_1 \times [\Delta_n, X] \to [\Delta_n, X]$$

by

$$H^*(t, f)(\sigma) := f(H(t, \sigma)).$$

Thus $r^*, \theta$ are homotopy inverses, as desired. □

## 6.3 Homotopy invariance of morphisms

Isomorphic objects give equivalent morphism spaces.

**Construction 6.3.1.** Let $X$ be an anima, and $x_0, x_1, x_0', x_1'$ four objects of $X$. Suppose we are given paths $p : x_0 \to x_1$ and $q : x_0' \to x_1'$.

We construct a homotopy class of maps

$$\mathrm{Mor}_X(x_0, x_0') \to \mathrm{Mor}_X(x_1, x_1')$$

as follows; we write $\varphi_{p,q}$ to denote a representative of this class.

By definition, $\mathrm{Mor}(x_i, x_i')$ is just the fiber of the Kan fibration

$$\theta : [\Delta_1, X] \to [\{0,1\}, X] = X \times X$$

above $(x_i, x_i')$. The paths $p, q$ in $X$ give us a path $(p, q) : (x_0, x_0') \to (x_1, x_1')$ in $X \times X$, and so Construction 5.3.1 gives us $\varphi_{p,q}$.

**Remark 6.3.2.** By Theorem 5.3.2, these maps $\varphi_{p,q}$ are always homotopy equivalences.

The results of Section 5.3, specialized to morphism spaces, produce the following results.

**Theorem 6.3.3.** *Let $f_0, f_1 : X \to Y$ be two homotopic functors between anima. Choose*

$$H : \Delta_1 \times X \to Y$$

*a homotopy between them.*

*Let $x, x' \in X_0$. Set $y_i = f_i(x), y_i' = f_i(x')$. The homotopy $H$ gives us paths $H_x : y_0 \to y_1, H_{x'} : y_0' \to y_1'$. The resulting diagram*

$$\begin{array}{ccc} \mathrm{Mor}_X(x,x') & \xrightarrow{f_0} & \mathrm{Mor}_Y(y_0,y_0') \\ & \searrow^{f_1} & \downarrow{\scriptstyle \varphi_{H_x,H_{x'}}} \\ & & \mathrm{Mor}_Y(y_1,y_1') \end{array}$$

*commutes up to homotopy.*

*Proof.* Immediate from Theorem 5.3.4, applied to the commutative diagram

$$\begin{array}{ccc} \Delta_1 \times [\Delta_1, X] & \xrightarrow{H^*} & [\Delta_1, Y] \\ \downarrow & & \downarrow \\ \Delta_1 \times [\{0,1\}, X] & \xrightarrow{H^*} & [\{0,1\}, Y], \end{array}$$

where

$$H^* : \Delta_1 \times [\Delta_1, X] \to [\Delta_1, Y]$$

is the map adjunct to

$$(t, f, s) \mapsto H(t, f(s)).$$

□

**Theorem 6.3.4.** *Let $f : X \to Y$ be a homotopy equivalence of anima. For any $x, x' \in X_0$, the induced functor*

$$f_* : \mathrm{Mor}_X(x,x') \to \mathrm{Mor}_Y(fx, fx')$$

*is also a homotopy equivalence of anima.*

*Proof.* Choose $g : Y \to X$ a homotopy inverse. Applying Corollary 5.3.5 to the commutative diagram

$$\begin{array}{ccc} [\Delta_1, X] & \xrightarrow{f} & [\Delta_1, Y] \\ \downarrow & & \downarrow \\ [\{0,1\}, X] & \xrightarrow{f} & [\{0,1\}, Y], \end{array}$$

we conclude (it is easy to construct the compatible homotopy inverses and homotopies using $g$ and the construction used in the proof of Theorem 6.3.3). □

## 6.4 2-morphisms

The morphisms of $\mathrm{Mor}_X(x, x')$ are called *2-morphisms* of $X$.

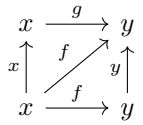
**Definition 6.4.1.** A *2-morphism* in an $\infty$-groupoid $X$, is the data of

1. a *source* $f \in X_1$,
2. a *target* $g \in X_1$ having the same endpoints $x, x'$ as $f$,
3. a 1-morphism of $\mathrm{Mor}_X(x, x')$ from $f$ to $g$.

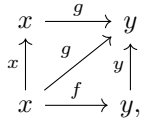
Given a fixed source and target $f, g : x \to x'$, we define

$$\mathrm{Mor}^2_X(f, g) := \mathrm{Mor}_{\mathrm{Mor}_X(x,x')}(f, g).$$

**Ideation 6.4.2.** Dealing with simplicial squares is a bit inconvenient; it is preferable to only have basic simplices. The goal of this section is to give us tools for dealing with 2-morphisms. As our first step, we note that there are some 'easy to specify' 2-morphisms: ones of the form



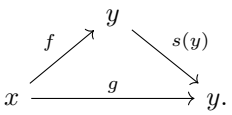

or



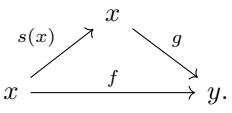

where one of the two 2-simplices making up the simplicial square is degenerate. This leads to the following definitions.

**Definition 6.4.3.** Suppose $X$ is an $\infty$-groupoid, and $f, g \in \mathrm{Mor}_X(x, y)$ are two 1-morphisms with the same endpoints.

A *left 2-morphism* from $f$ to $g$ is a 2-simplex $\sigma \in X_2$ with boundary




Similarly, a *right 2-morphism* from $f$ to $g$ is a 2-simplex with boundary



The $\infty$-*groupoid* of left 2-morphisms from $f$ to $g$ is the fiber of

$$[\Delta_2, X] \to [\partial\Delta_2, X]$$

above the boundary with 01 edge $f$, 02 edge $g$, and 12 edge $s(y)$. Similarly we can define the $\infty$-groupoid of right 2-morphisms from $f$ to $g$.

We write $\mathrm{Mor}_X^L(f,g)$ for the $\infty$-groupoid of left 2-morphisms and $\mathrm{Mor}_X^R(f,g)$ for the $\infty$-groupoid of right 2-morphisms.

**Remark 6.4.4.** The anima $\mathrm{Mor}_X^2(f,g)$ is the morally correct anima of 2-morphisms, but it is very inconvenient to use in practice; the left and right mapping spaces are much simpler technically.

Fortunately for us, the left mapping space, right mapping space, and $\mathrm{Mor}_X^2(f,g)$ are all equivalent.

**Construction 6.4.5.** We construct here functors

$$\mathrm{Mor}_X^L(f,g) \to \mathrm{Mor}_X^2(f,g),$$
$$\mathrm{Mor}_X^R(f,g) \to \mathrm{Mor}_X^2(f,g).$$

To start, put

$$\varphi_L : [\Delta_2, X] \to [\Delta_1 \times \Delta_1, X]$$

the following function: on $T$-points, $\varphi_L$ is (after currying in the obvious way) the function

$$\mathrm{Mor}_{\mathrm{sSet}}(\Delta_2, [T,X]) \to \mathrm{Mor}_{\mathrm{sSet}}(\Delta_1 \times \Delta_1, [T,X])$$

sending $\sigma \in [T,X]_2$ to the map

$$\varphi_L(\sigma) : \Delta_1 \times \Delta_1 \to [T,X]$$

given by

$$\begin{array}{ccc} \sigma(0) & \xrightarrow{\sigma_{02}} & \sigma(2) \\ {\scriptstyle \sigma(0)}\uparrow & \nearrow_{\sigma_{02}} & \uparrow{\scriptstyle \sigma_{12}} \\ \sigma(0) & \xrightarrow[\sigma_{01}]{} & \sigma(1), \end{array}$$

where the lower 2-simplex is filled in with $\sigma$ and the upper one is filled in with the degenerate 2-simplex $s^0(\sigma_{02})$.

As $\mathrm{Mor}_X^L(f,g), \mathrm{Mor}_X^2(f,g)$ are simplicial subsets of $[\Delta_2, X], [\Delta_1 \times \Delta_1, X]$ (resp.), it is easy to see that $\varphi_L$ induces a functor

$$\mathrm{Mor}_X^L(f,g) \to \mathrm{Mor}_X^2(f,g).$$

A similar construction works for $\mathrm{Mor}_X^R(f,g) \to \mathrm{Mor}_X^2(f,g)$.

**Theorem 6.4.6.** *Let $X$ be an $\infty$-groupoid, and $f, g : x \to x'$ two 1-morphisms with the same source and target. Then the functors*

$$\varphi_L : \mathrm{Mor}_X^L(f,g) \to \mathrm{Mor}_X^2(f,g),$$
$$\varphi_R : \mathrm{Mor}_X^R(f,g) \to \mathrm{Mor}_X^2(f,g)$$

*of Construction 6.4.5 are both homotopy equivalences of anima.*

Theorem 6.4.6 will follow as a special case of the upcoming Theorem 6.5.6 (or, more precisely, the claim about left mapping spaces will follow from Theorem 6.5.6 – we leave the right mapping space claim to the reader; as a hint, the 'opposite' functor introduced in the appendix can be helpful).

## 6.5 $n$-morphisms in general

In the previous section, we studied 2-morphisms in detail. The obvious next step is to replace 2 by $n$.

**Definition 6.5.1.** Let $X$ be an anima. We inductively define $n$-morphisms as follows.

A *0-morphism* is an object of $X$.

Given two $n$-morphisms $\eta, \eta'$, with the same source $f$ and target $f'$, we define

$$\mathrm{Mor}_X^{n+1}(\eta, \eta') := \mathrm{Mor}_{\mathrm{Mor}_X^n(f,f')}(\eta, \eta').$$

The objects of those anima are said to be $(n+1)$-*morphisms* of $X$, with source $\eta$ and target $\eta'$.

**Remark 6.5.2.** Suppose $f_0, f_0' \in X_0$ are objects, and $f_i, f_i' : f_{i-1} \to f_{i-1}'$ is a sequence of parallel morphisms.

The 2-morphisms of $\mathrm{Mor}_X^n(f_{n-1}, f_{n-1}')$ are easy to describe in terms of $X$:

$$\mathrm{Mor}^2_{\mathrm{Mor}_X^n(f_{n-1},f_{n-1}')}(f_{n+1}, f_{n+1}') = \mathrm{Mor}_{\mathrm{Mor}_X^{n+1}(f_n,f_n')}(f_{n+1}, f_{n+1}') = \mathrm{Mor}_X^{n+2}(f_{n+1}, f_{n+1}').$$

Iterating this procedure, we find that there's a canonical *isomorphism* of simplicial sets

$$\mathrm{Mor}^i_{\mathrm{Mor}_X^j(f_{j-1},f_{j-1}')}(f_{i+j-1}, f_{i+j-1}') = \mathrm{Mor}_X^{i+j}(f_{i+j-1}, f_{i+j-1}')$$

for all $i, j$. To do this formally, one should induct on $i$, so that it makes sense to consider $i+j-1$ morphisms of $X$ as $(i-1)$ morphisms of $\mathrm{Mor}_X^j$.

As with 2-morphisms, $n$-morphisms as defined above are a bit unwieldy. It is much more convenient to model them with maps $\Delta_n \to X$, leading to the notion of a *left* $n$-morphism.

### 6.5.1 Left morphisms

There are some annoying boundary conditions to get right in the definition of a left $n$-morphism, so for motivation we start by doing the case $n = 3$ explicitly.

**Ideation 6.5.3** (Left 3-morphisms)**.** Let $X$ be an anima. Fix $f, g : x \to y$ two parallel 1-morphisms.

Let $\sigma, \sigma' \in X_2$ be two left 2-morphisms, both going from $f$ to $g$. What sort of object should a left 3-morphism between them be?

We want it to be some kind of 3-simplex. In a left 2-morphism, the source $f$ is opposite to the vertex 2, and the target $g$ is opposite to the vertex 1. This suggests we look at 3-simplices $\tau$ such that $d^3(\tau) = \sigma$ is our source and $d^2(\tau) = \sigma'$ is our target.

Thus the 012 face is $\sigma$, and the 013 face is $\sigma'$. This means that

1. the 01 edge is $f$,
2. the 02 edge is $g$,
3. the 03 edge is $g$,
4. the 12 edge is $s(y)$,
5. the 13 edge is $s(y)$.

At the moment, the 23 edge is undetermined, but it needs to go from $y$ to $y$, and (to fill the 123 face) it needs to be homotopic to $\mathrm{id}_y$. Thus we just define the 23 edge to be $s(y)$, and let the 123 face of $\tau$ be $s(y)$. Thus

$$d^0(\tau) = s(y) = s^1(d^0(\sigma)).$$

Lastly, we need to understand the 023 face of $\tau$. This has edges $g$, $s(y)$, and $g$, so it makes the most sense to require $d^1(\tau) = s^1(g) = s^1(d^1(\sigma))$.

**Definition 6.5.4.** Let $X$ be an anima. We inductively define *left $n$-morphisms* as follows.

A left 0-morphism is an object of $X$. A left 1-morphism is a morphism of $X$.

For $n \geq 1$, a left $(n+1)$-morphism is some $\eta \in X_{n+1}$ so that

1. $f := d^{n+1}(\eta)$ is a left $n$-morphism,
2. $g := d^n(\eta)$ is a left $n$-morphism,
3. $d^i(f) = d^i(g)$ for every $0 \leq i \leq n$,
4. for each $0 \leq i \leq n-1$, we have $d^i(\eta) = s^{n-1}(d^i(f))$.

**Definition 6.5.5.** We say two left $n$-morphisms $f, g$ are *parallel* if $d^i(f) = d^i(g)$ for every $0 \leq i \leq n$.

Given two parallel left $n$-morphisms $f, g$, we define an $\infty$-groupoid

$$\mathrm{Mor}_X^{(n+1),L}(f, g)$$

as the fiber of the Kan fibration

$$[\Delta_{n+1}, X] \to [\partial\Delta_{n+1}, X]$$

over the map $\partial\Delta_{n+1} \to X$ whose $(n+1)$-face is $f$, whose $n$-face is $g$, and whose $i$-face for $0 \leq i \leq n-1$ is given by $s^{n-1}(d^i f)$.

As evidence that Definition 6.5.4 is reasonable, we will prove the following.

**Theorem 6.5.6.** *Let $X$ be an anima. Fix $\eta, \eta' \in X_n$ two parallel left $n$-morphisms, with source $f$ and target $f'$. Then there is a homotopy equivalence*

$$\mathrm{Mor}_X^{(n+1),L}(\eta, \eta') \simeq \mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(f,f')}(\eta, \eta').$$

**Remark 6.5.7.** Iteratively applying Theorem 6.5.6, one can compare left $n$-morphisms to $n$-morphisms. We will not elaborate on this point as we won't need this precise claim, but one should understand that left $n$-morphisms are just as good as usual $n$-morphisms; however, see Section 7.6 for a (somewhat limited) implementation of this iteration idea.

We prove Theorem 6.5.6 in several steps. The basic idea is as follows.

**Ideation 6.5.8.** Fix $\eta, \eta'$ two parallel left $n$-morphisms, as in Theorem 6.5.6. The anima $\mathrm{Mor}_X^{(n+1),L}(\eta, \eta')$ is a particular fiber of

$$[\Delta_{n+1}, X] \to [\partial\Delta_{n+1}, X].$$

Similarly, $\mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(f,f')}(\eta, \eta')$ is a particular fiber of

$$[\Delta_1 \times \Delta_n, X] \to [A_n, X],$$

where

$$A_n := (\Delta_1 \times \partial\Delta_n) \coprod_{\partial\Delta_1 \times \partial\Delta_n} (\partial\Delta_1 \times \Delta_n).$$

To relate these two fibers, we start by constructing a functor

$$[\Delta_{n+1}, X] \to [\Delta_1 \times \Delta_n, X].$$

The easiest way to find such a functor is to create a map

$$r_n : \Delta_1 \times \Delta_n \to \Delta_{n+1}$$

which we can precompose with. In the $n = 1$ case, we used a map which collapsed the simplicial square onto its bottom simplex. We will construct a similar contraction in general, having the property that $r_n$ sends $A_n$ to $\partial\Delta_{n+1}$. This will induce a commutative diagram

$$\begin{array}{ccc} [\Delta_{n+1}, X] & \xrightarrow{r_n^*} & [\Delta_1 \times \Delta_n, X] \\ \downarrow & & \downarrow \\ [\partial\Delta_{n+1}, X] & \xrightarrow{r_n^*} & [A_n, X]. \end{array}$$

This induces a map of the fibers, and we will prove that map is a homotopy equivalence by using that $r_n$ is. In fact, $r_n$ is a very simple type of homotopy equivalence – a retraction – which makes our argument easier.

Our proof will go in three parts:

1. first we compare fibers between retractions in general,
2. then we construct $r_n$,
3. then we conclude.

There is a small technical issue which we encounter when constructing $r_n$, but we will cross that bridge when we come to it.

### 6.5.2 Fibers and retractions

We start by explaining why constructing the retraction alluded to in Ideation 6.5.8 is enough to conclude.

**Lemma 6.5.9.** *Let $i : A \hookrightarrow B$ be an inclusion of simplicial sets, and let $A', B'$ be simplicial subsets of $A, B$ such that $i$ restricts to an inclusion $i' : A' \hookrightarrow B'$. Suppose we have maps*

$$r : B \to A,$$

$$H : B \times \Delta_1 \to B,$$

*obeying the following properties:*

1. $r \circ i = \mathrm{id}_A$,
2. $r(B') \subseteq A'$,

*3. $H$ is a homotopy from $ir$ to $\mathrm{id}_B$,*

*4. $H|_{B'\times\Delta_1}$ factors through $B'$,*

*5. the resulting map $H' : B' \times \Delta_1 \to B'$ is a homotopy from $i'r'$ to $\mathrm{id}_{B'}$.*

*Then, for any anima $X$, and any point $a' \in [A', X]_0$, the commutative diagram*

$$\begin{array}{ccc} [A,X] & \xrightarrow{r^*} & [B,X] \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ [A',X] & \xrightarrow{(r')^*} & [B',X] \end{array}$$

*induces a* homotopy equivalence

$$\alpha : \theta_A^{-1}(a') \to \theta_B^{-1}(a' \circ r').$$

*Proof.* The maps $\theta_A, \theta_B$ are Kan fibrations. The maps $r^*, i^*$ are homotopy inverses, as are $(r')^*, (i')^*$; as $ri = \mathrm{id}_A, r'i' = \mathrm{id}_{A'}$, and by the hypotheses we impose on $B$, Corollary 5.3.5 then immediately implies our claim. □

### 6.5.3 Constructing $r_n$ and proving a map anodyne

There are $n+1$ nondegenerate $(n+1)$-simplices of $\Delta_1 \times \Delta_n$. We label them $\sigma_0, ..., \sigma_n$, with $\sigma_i$ being the simplex with vertices

$$(0,0), (0,1), ..., (0,i), (1,i), (1,i+1), ..., (1,n).$$

Let

$$i_n : \Delta_{n+1} \hookrightarrow \Delta_1 \times \Delta_n$$

be the inclusion onto $\sigma_n$.

We would like to use this $i_n$ as the inclusion in Lemma 6.5.9, but there is a big problem: $i_n(\partial\Delta_{n+1})$ is *not* contained in $A_n$. To fix this, we first replace $A_n$ by a larger subset of $\Delta_1 \times \Delta_n$, in such a way that the fiber of

$$[\Delta_1 \times \Delta_n, X] \to [A_n, X]$$

is unchanged up to homotopy after replacing $A_n$ with its enlargement.

**Lemma 6.5.10.** *Let $A \subseteq A' \subseteq B$ be inclusions of simplicial sets such that $A \hookrightarrow A'$ is strongly anodyne. Choose $X$ an anima, and $a' : A' \to X$ a map. Set $a := a'|_A$.*

*Consider the Kan fibrations*

$$\theta : [B, X] \to [A, X]$$

*and*

$$\theta' : [B, X] \to [A', X].$$

*The inclusion*

$$i : (\theta')^{-1}(a') \hookrightarrow \theta^{-1}(a)$$

*is a homotopy equivalence.*

*Proof.* As $A \hookrightarrow A'$ is strongly anodyne, the restriction map

$$\varphi : [A', X] \to [A, X]$$

is a trivial Kan fibration.

In particular, its base changes are trivial Kan fibrations, and so $\varphi^{-1}(a) \to \mathrm{pt}$ is a trivial Kan fibration. Thus we may find a map

$$H : \varphi^{-1}(a) \times \Delta_1 \to \varphi^{-1}(a)$$

whose restriction to time 0 is id, whose restriction to time 1 is the constant map with value $a'$, and whose restriction to $\{a'\} \times \Delta_1$ is again the constant map with value $a'$.

The base change of $[B, X] \to [A', X]$ along $\varphi^{-1}(a) \hookrightarrow [A', X]$ is the projection

$$\pi : \theta^{-1}(a) \to \varphi^{-1}(a),$$

which is therefore a Kan fibration.

As

$$(\theta^{-1}(a) \times \{0\}) \coprod_{(\theta')^{-1}(a') \times \{0\}} ((\theta')^{-1}(a') \times \Delta_1) \hookrightarrow \theta^{-1}(a) \times \Delta_1$$

is anodyne by Theorem 4.8.8, we may lift $H \circ (\pi \times \mathrm{id})$ to a map

$$\tilde{H} : \theta^{-1}(a) \times \Delta_1 \to \theta^{-1}(a)$$

such that $\tilde{H}_0 = \mathrm{id}$ and such that $\tilde{H}|_{(\theta')^{-1}(a') \times \Delta_1}$ is the constant homotopy from id to id. By the compatibility of $\tilde{H}$ over $H \circ (\pi \times \mathrm{id})$, we deduce that $\tilde{H}_1$ factors through $(\theta')^{-1}(a')$.

Define

$$r : \theta^{-1}(a) \to (\theta')^{-1}(a')$$

the restriction of $\tilde{H}$ to $\theta^{-1}(a) \times \{1\}$. As we required $\tilde{H}$ to be constant on $(\theta')^{-1}(a')$, the composition $r \circ i$ is identity. Moreover, $ir \simeq \mathrm{id}$, as $\tilde{H}$ itself gives us such a homotopy: at time 0 it is identity, and at time 1 it is $ir$. Thus $i$ is a homotopy equivalence, with $r$ a homotopy inverse. □

Let $B_n \subseteq \Delta_1 \times \Delta_n$ be the simplicial subset obtained as the union

$$B_n := A_n \cup \bigcup_{i=0}^{n-1} \sigma_i.$$

Observe that $i_n(\partial\Delta_{n+1}) \subseteq B_n$, so as long as we can prove $A_n \hookrightarrow B_n$ is strongly anodyne, we're happy. And indeed, we can.

**Lemma 6.5.11.** *The inclusion $A_n \hookrightarrow B_n$ is strongly anodyne.*

*Proof.* Set $B_n^{-1} = A_n$, and

$$B_n^i = A_n \cup \sigma_0 \cup \cdots \cup \sigma_i.$$

It is enough to prove that each inclusion $B_n^{i-1} \hookrightarrow B_n^i$ is strongly anodyne. But observe that we have a pushout diagram

$$\begin{array}{ccc} \Lambda^{i+1}_{n+1} & \longrightarrow & \Delta_{n+1} \\ \downarrow & & \downarrow \\ B^{i-1}_n & \longrightarrow & B^i_n, \end{array}$$

and as pushouts of strongly anodyne maps are strongly anodyne, our inclusion is strongly anodyne, as desired. □

We are now in a good position to build our retraction

$$r_n : \Delta_1 \times \Delta_n \to \Delta_{n+1}$$

to $i_n$. Let $r_n$ be the morphism of simplicial sets induced by the monotone map

$$r_n(0, i) = i,$$

$$r_n(1, i) = \begin{cases} i & i < n, \\ n+1 & i = n. \end{cases}$$

To end this subsection, we verify that $r_n, i_n$ obey the compatibilities needed to apply Lemma 6.5.9.

**Lemma 6.5.12.** *The map $r_n$ sends $B_n$ into $\partial\Delta_{n+1}$. Let $r'_n : B_n \to \partial\Delta_{n+1}$ be the resulting restriction. Similarly, $i_n$ gives a map $i'_n : \partial\Delta_{n+1} \hookrightarrow B_n$.*

*The composition $r_n \circ i_n$ is identity, and there exists a homotopy*

$$H : \Delta_1 \times (\Delta_1 \times \Delta_n) \to \Delta_1 \times \Delta_n$$

*from $i_n \circ r_n$ to* id. *Moreover, this homotopy can be chosen such that its restriction to $\Delta_1 \times B_n$ factors through $B_n$, and the resulting map is a homotopy from $i'_n r'_n$ to* id.

*Proof.* To see $r_n(B_n) \subseteq \partial\Delta_{n+1}$, it suffices to prove

$$r_n(\sigma_i) \subseteq \partial\Delta_{n+1}$$

for each $0 \leq i < n$, and that $r_n(A_n) \subseteq \partial\Delta_{n+1}$. For the simplex claim, just observe that $r_n(\sigma_i)$ is the $(n+1)$-simplex of $\Delta_{n+1}$ with vertices

$$0, 1, ..., i, i, i+1, ..., n-1, n+1,$$

and in particular is *degenerate*, implying it lies in the boundary. For the claim about $A_n$, it suffices to check

$$r_n(\Delta_1 \times \partial\Delta_n) \subseteq \partial\Delta_{n+1},$$
$$r_n(\partial\Delta_1 \times \Delta_n) \subseteq \partial\Delta_{n+1},$$

and both of these claims are again easy by vertex checking. The claim about $i'_n$ is similar.

To see $r_n i_n = \mathrm{id}$, observe that when $j < n+1$, we have

$$r_n(i_n(j)) = r_n(0, j) = j,$$

and when $j = n+1$, we have

$$r_n(i_n(n+1)) = r_n(1,n) = n+1.$$

Hence $r_n i_n = \mathrm{id}$.

Next, the homotopy. Define

$$H : \Delta_1 \times (\Delta_1 \times \Delta_n) \to \Delta_1 \times \Delta_n$$

by

$$H(0,j,k) = i_n r_n(j,k),$$
$$H(1,j,k) = (j,k).$$

To see this is monotone, it suffices to prove $i_n r_n(j,k) \leq (j,k)$. When $j = 0$, this is clear (as both are equal), and when $j = 1$, either $k < n$ and so

$$i_n r_n(1,k) = i_n(k) = (0,k) < (1,k),$$

or $k = n$ and

$$i_n r_n(1,n) = i_n(n+1) = (1,n).$$

To conclude, we need to show $H(\Delta_1 \times B_n) \subseteq B_n$.

By construction of $B_n$, the only non-degenerate simplex of $\Delta_1 \times \Delta_n$ which is not contained in $B_n$ is $\sigma_n$, so it suffices to show that $H$ does not map any $(n+1)$-simplex of $\Delta_1 \times B_n$ to a $\sigma_n$.

And indeed, take $(\sigma, \tau) \in (\Delta_1 \times (\Delta_1 \times \Delta_n))_{n+1}$ such that $H(\sigma,\tau) = \sigma_n$.

Observe that $(0,n)$ is the $n^{\text{th}}$ vertex of $\sigma_n$ and the only vertex of $\Delta_1 \times \Delta_1 \times \Delta_n$ which $H$ sends to $(0,n)$ are $(1,(0,n))$ and $(0,(0,n))$. Thus the $n^{\text{th}}$ vertex of $(\sigma,\tau)$ must be one of these; say it is $(i_n,(0,n))$.

In particular, the $n^{\text{th}}$ vertex of $\tau$ must be $(0,n)$. For each $k \leq n$, the $k^{\text{th}}$ vertex of $(\sigma,\tau)$ is thus less than $(i_n,(0,n))$, and so of the form $(i_k,(0,j_k))$, where $i_k \leq i_n, j_k \leq n$. In particular,

$$H(i_k,(0,j_k)) = (0,j_k),$$

and so $j_k$ must be equal to $k$, as the $k^{\text{th}}$ vertex of $\sigma_n$ is $(0,k)$.

Similarly, we see $\tau$ contains $(1,n)$ as its final vertex, as we know its final vertex is sent by $H$ to $(1,n)$. Thus $\tau = \sigma_n$, and hence $\tau \notin B_n$, as desired. □

### 6.5.4 Finishing our argument

We can now finish our proof of Theorem 6.5.6. Take $X$ an anima, and $\eta, \eta' \in X_n$ two parallel left $n$-morphisms, with source $f$ and target $f'$. Recall that Theorem 6.5.6 asserts the existence of a homotopy equivalence

$$\mathrm{Mor}_X^{(n+1),L}(\eta,\eta') \simeq \mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(f,f')}(\eta,\eta').$$

*Proof of Theorem 6.5.6.* We use the notation $i_n, r_n, A_n, B_n$ of the previous subsections.

The anima $\mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(f,f')}(\eta,\eta')$ is defined as the fiber of

$$\theta : [\Delta_1 \times \Delta_n, X] \to [A_n, X]$$

above the map

$$a : A_n \to X$$

given as follows: on $0 \times \Delta_n$, we have $a = \eta$; on $1 \times \Delta_n$, we have $a = \eta'$; and on $\Delta_1 \times \partial\Delta_n$, we have $a$ is the composite

$$\Delta_1 \times \partial\Delta_n \to \partial\Delta_n \xrightarrow{\bar{a}} X,$$

where $\bar{a} : \partial\Delta_n \to X$ is the restriction of either $\eta$ or $\eta'$ to $\partial\Delta_n$.

Let $c : \partial\Delta_{n+1} \to X$ be the map so that $\mathrm{Mor}_X^{n+1,L}(\eta, \eta')$ is the fiber of

$$[\Delta_{n+1}, X] \to [\partial\Delta_{n+1}, X]$$

above $c$.

Define $b : B_n \to X$ as the composite map $c \circ r'_n$. Then Lemma 6.5.10 tells us that $\theta^{-1}(a) \simeq (\theta')^{-1}(b)$, where $\theta'$ is the restriction map

$$\theta' : [\Delta_1 \times \Delta_n, X] \to [B_n, X].$$

Lemma 6.5.9 and Lemma 6.5.12 then allow us to conclude. □

In fact, our proof of Theorem 6.5.6 proved slightly more than was stated on the tin: we constructed a *particular* homotopy equivalence between the two anima. We isolate this homotopy equivalence for later use.

**Construction 6.5.13.** Let $X$ be an anima, and $\eta, \eta' \in X_n$ two parallel left $n$-morphisms with source $f$ and target $f'$. We write

$$\Phi_{\eta,\eta'} : \mathrm{Mor}_X^{n+1,L}(\eta, \eta') \to \mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(f,f')}(\eta, \eta')$$

for the homotopy equivalence constructed in the proof of Theorem 6.5.6. More precisely, using the notation of the proof of Theorem 6.5.6, and letting

$$\varphi : [\Delta_{n+1}, X] \to [\partial\Delta_{n+1}, X]$$

denote the restriction map, our map $\Phi_{\eta,\eta'}$ is the composition

$$\mathrm{Mor}_X^{n+1,L}(\eta, \eta') = \varphi^{-1}(c) \xrightarrow{r_n^*} (\theta')^{-1}(b) \hookrightarrow \theta^{-1}(a) = \mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(f,f')}(\eta, \eta').$$

## 6.6 Composition of $n$-morphisms

Let $X$ be an $\infty$-groupoid. How do we compose $n$-morphisms in $X$?

### 6.6.1 Composition of 1-morphisms

We start with better understanding composition of 1-morphisms.

**Construction 6.6.1** (Composition)**.** We construct a composition function

$$\circ : \pi_0 \operatorname{Mor}_X(x, x') \times \pi_0 \operatorname{Mor}_X(x', x'') \to \pi_0 \operatorname{Mor}_X(x, x'') \tag{6.6.2}$$

as follows.

Define

$$S_{x,x',x''} := [\Delta_2, X] \times_{X^3} \{(x, x', x'')\}$$

the space of 2-simplices with prescribed vertices. Similarly we define

$$T_{x,x',x''} := [\Lambda^1_2, X] \times_{X^3} \{(x, x', x'')\}.$$

As $\Lambda^1_2$ is the coproduct of two copies of $\Delta_1$ joined at opposite endpoints, we find

$$T_{x,x',x''} \simeq \operatorname{Mor}_X(x, x') \times \operatorname{Mor}_X(x', x'').$$

As $\Lambda^1_2 \to \Delta_2$ is strongly anodyne,

$$[\Delta_2, X] \to [\Lambda^1_2, X]$$

is a trivial Kan fibration. Thus the base change

$$p : S_{x,x',x''} \to T_{x,x',x''}$$

is also a trivial Kan fibration, and in particular a homotopy equivalence.

Let

$$q : S_{x,x',x''} \to \operatorname{Mor}_X(x, x'')$$

be restriction to the 02-edge. Then we define

$$[g] \circ [f] := (\pi_0(q) \circ \pi_0(p)^{-1})([f], [g]).$$

Recall that, in Definition 4.4.4, we gave a different definition of composition. We show now that these two notions agree.

**Proposition 6.6.3.** *Let $X$ be an $\infty$-groupoid. Take arrows $f : x \to x', g : x' \to x'', h : x \to x''$ in $X$. Then $h$ is a composition of $g$ and $f$ (in the sense of Definition 4.4.4) if and only if $[h] = [g] \circ [f]$ under (6.6.2).*

*Proof.* We copy the notation of Construction 6.6.1. Suppose first that $h$ is a composition of $g$ and $f$. Then there exists a point $\sigma \in (S_{x,x',x''})_0$ such that $p(\sigma) = (f, g)$ and $q(\sigma) = h$. Thus $[g] \circ [f] = [h]$.

Suppose next that $[g] \circ [f] = [h]$. As $p : S_{x,x',x''} \to T_{x,x',x''}$ is a trivial Kan fibration, it admits a section $s : T_{x,x',x''} \to S_{x,x',x''}$.

We thus have that $h' := q(s(f, g))$ is homotopic to $h$ in $\operatorname{Mor}_X(x, x'')$. The inclusion

$$(\partial\Delta_2 \times \Delta_1) \coprod_{\partial\Delta_2 \times \{0\}} \Delta_2 \times \{0\} \hookrightarrow \Delta_2 \times \Delta_1$$

is anodyne, and so we may solve an extension problem to find a 2-simplex $\sigma \in X_2$ with edges $f, g$, and $h$, proving $h$ is a composition of $f$ and $g$, as desired. Indeed, on our first factor $\partial\Delta_2 \times \Delta_1$ we use the homotopy from $(f, g, h')$ to $(f, g, h)$ (glue the constant homotopy $(f, g) \simeq (f, g)$ with a path in $\operatorname{Mor}_X(x, x'')$ from $h'$ to $h$) and on the factor $\Delta_2 \times \{0\}$ we use the 2-simplex $s(f, g)$. □

**Notation 6.6.4.** We will write

$$g \circ f \simeq h$$

to denote $[g] \circ [f] = [h]$.

Composition is associative.

**Proposition 6.6.5** (Composition is associative)**.** *Let $X$ be an $\infty$-groupoid, and $f : x \to y, g : y \to z, h : z \to w$ three morphisms of $X$. Then*

$$h \circ (g \circ f) \simeq (h \circ g) \circ f.$$

*Proof.* Let $\sigma \in X_2$ be some 2-simplex with 01 edge $f$ and 12 edge $g$. Similarly, let $\tau$ be some 2-simplex with 01 edge $g$ and 12 edge $h$.

Finally, choose $\tau'$ some 2-simplex with 01 edge $d^1(\sigma)$ and 12 edge $h$. Then $\sigma, \tau, \tau'$ glue together to form a $\Lambda^2_3$-horn (with $\sigma$ the 012 face, $\tau$ the 123 face, and $\tau'$ the 023 face); fill this horn. Its 013 face will be some 2-simplex of $X$ witnessing that

$$d^1(\tau) \circ f \simeq d^1(\tau').$$

But $\tau'$ witnesses that $d^1(\tau') \simeq h \circ d^1(\sigma)$, and $\sigma, \tau$ witness

$$d^1(\sigma) \simeq g \circ f,$$

$$d^1(\tau) \simeq h \circ g.$$

Putting this all together, we find

$$(h \circ g) \circ f \simeq h \circ (g \circ f),$$

as desired. □

Similarly, composition is unital, and morphisms have inverses.

**Proposition 6.6.6.** *Let $X$ be an $\infty$-groupoid and $f : x \to y$ a morphism. Then*

$$s(y) \circ f \simeq f \circ s(x) \simeq f,$$

*and there exists a morphism $f^{-1} : y \to x$ (not necessarily unique) so that*

$$f \circ f^{-1} \simeq s(y),$$

$$f^{-1} \circ f \simeq s(x).$$

**Remark 6.6.7.** The connected component $[f^{-1}]$ of $\pi_0 \operatorname{Mor}_X(y, x)$ *is* unique, by associativity of composition.

*Proof.* For the first claim, use the two degeneracies associated to $f$. For the second claim, solving $\Lambda^2_2$ and $\Lambda^0_2$ horn filling problems shows that $f$ always has a left inverse and a right inverse; associativity of composition implies the left and right inverse are equivalent, as desired. □

### 6.6.2 The fundamental groupoid

In particular, we can construct a canonical 1-groupoid associated to an $\infty$-groupoid.

**Construction 6.6.8** (The fundamental groupoid)**.** Let $X$ be an $\infty$-groupoid. The *fundamental groupoid* of $X$ is the groupoid $\pi_{\leq 1}(X)$ where

1. objects are objects of $X$,
2. morphisms $x \to x'$ are classes in $\pi_0 \operatorname{Mor}_X(x, x')$,
3. composition is defined using Construction 6.6.1.

This defines a groupoid by Proposition 6.6.5 and Proposition 6.6.6.

**Remark 6.6.9.** It is essential to define morphisms to be

$$\pi_0(\operatorname{Mor}_X(x, x'))$$

instead of $\operatorname{Mor}_X(x, x')_0$. Without taking this quotient by path equivalence, associativity would fail in general, and composition would depend on an arbitrary choice of a section to

$$[\Delta_2, X] \to [\Lambda^1_2, X].$$

**Notation 6.6.10.** For $X$ an anima and $x \in X_0$, we define

$$\pi_1(X, x) := \operatorname{Aut}_{\pi_{\leq 1}(X)}(x).$$

We refer to $\pi_1(X, x)$ as the *fundamental group* of $(X, x)$.

**Construction 6.6.11** (Functoriality of the fundamental groupoid)**.** Let $f : X \to Y$ be a morphism of Kan complexes. We define

$$\pi_{\leq 1}(f) : \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$$

the functor

$$\pi_{\leq 1}(f)(x) = f(x),$$
$$\pi_{\leq 1}(f)([e]) = [f(e)].$$

More precisely, this map on morphisms is obtained by taking $\pi_0$ of the natural map

$$f_* : \operatorname{Mor}_X(x, x') \to \operatorname{Mor}_Y(fx, fx').$$

We have seen another way to turn simplicial sets into categories: the categorical realization functor. It turns out that the categorical realization functor and $\pi_{\leq 1}$ coincide.

**Lemma 6.6.12.** *Let $X$ be an $\infty$-groupoid. Define a morphism of simplicial sets*

$$\varphi : X \to N(\pi_{\leq 1} X)$$

*as follows:*

1. *on 0-simplices, $\varphi_0(x) = x$,*

2. *on 1-simplices,* $\varphi_1(e) = [e]$,

3. *for* $\sigma \in X_n$ *whose vertices are* $x_0, x_1, ..., x_n$ *and where the* $i \to (i+1)$ *edge is* $e_i$*, we define*

$$\varphi_n(\sigma) = x_0 \xrightarrow{[e_0]} x_1 \xrightarrow{[e_1]} \cdots \xrightarrow{[e_{n-1}]} x_n.$$

*This morphism is simplicial as nerves of categories are 2-coskeletal.*

*Then the adjunct functor*

$$F : \mathfrak{C}[X] \to \pi_{\leq 1}(X)$$

*is an equivalence of categories; in fact, it is even an* isomorphism *of categories.*

*Proof.* Let

$$u_X : X \to N(\mathfrak{C}[X])$$

be the unit of the adjunction between categorical realization and nerves. Define

$$G : \pi_{\leq 1}(X) \to \mathfrak{C}[X]$$

by

$$G(x) := u_X(x)$$

on objects, and

$$G([a]) := u_X(a)$$

on morphisms; to see this is well-defined, note that if $[a] = [a']$, then we may find a map $\Delta_1 \times \Delta_1 \to X$ with top edge $a$, bottom edge $a'$, and side edges $\mathrm{id}$. Composing with $u_X$, we find a commutative diagram

$$\begin{array}{ccc} u_X(x) & \xrightarrow{u_X(a)} & u_X(y) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ u_X(x) & \xrightarrow{u_X(a')} & u_X(y) \end{array}$$

in $\mathfrak{C}[X]$, forcing $u_X(a) = u_X(a')$.

Similarly we can check $G$ is a functor using 2-simplices.

As $F$ is the adjunct to $\varphi$, we have

$$N(F) \circ u_X = \varphi.$$

Thus

$$F(G(x)) = F(u_X(x)) = \varphi_0(x) = x$$

and

$$F(G([e])) = F(u_X(e)) = [e].$$

So $F \circ G = \mathrm{id}$.

Similarly, the functor

$$GF : \mathfrak{C}[X] \to \mathfrak{C}[X]$$

is adjunct to the simplicial set map

$$X \xrightarrow{u_X} N(\mathfrak{C}[X]) \xrightarrow{N(F)} N(\pi_{\leq 1}X) \xrightarrow{N(G)} N(\mathfrak{C}[X]).$$

As $N(F) \circ u_X = \varphi$, this is just $N(G) \circ \varphi$.

On 0-simplices,

$$N(G)(\varphi(x)) = G(x) = u_X(x),$$

and on 1-simplices,

$$N(G)(\varphi(e)) = G([e]) = u_X(e).$$

As the higher simplices of the nerve of a category are determined by their 1-skeletons, it follows that

$$N(G) \circ \varphi = u_X.$$

Thus $GF$ and $\mathrm{id}_{\mathfrak{C}[X]}$ are both adjunct to $u_X$, implying they are equal; hence $F$ is an isomorphism of categories, as desired. □

**Corollary 6.6.13.** *The functor*

$$\pi_{\leq 1} : \mathrm{Kan} \to \mathrm{Gpd}$$

*is the left adjoint to*

$$N : \mathrm{Gpd} \to \mathrm{Kan}\,.$$

*Proof.* Immediate from the adjunction between nerves and categorical realization. □

It is useful to know that $\pi_{\leq 1}$ is also compatible with the 2-morphisms in Gpd .

**Construction 6.6.14** (Homotopy invariance of $\pi_{\leq 1}$)**.** Assume $f, g : X \to Y$ are two homotopic functors between anima. Choose

$$H : X \times \Delta_1 \to Y$$

a homotopy from $f$ to $g$.

Then we construct a natural isomorphism

$$\eta_H : \pi_{\leq 1}(f) \to \pi_{\leq 1}(g)$$

associated to $H$ as follows.

Each object $x \in X_0$ of $\pi_{\leq 1}(X)$ gives us a map $H_x : \Delta_1 \to Y$, which defines a morphism $\eta_H(x) : f(x) \to g(x)$.

To see this is a natural transformation, we need the squares

$$\begin{array}{ccc} f(x) & \xrightarrow{\eta_H(x)} & g(x) \\ \downarrow{\scriptstyle f(p)} & & \downarrow{\scriptstyle g(p)} \\ f(x') & \xrightarrow{\eta_H(x')} & g(x') \end{array}$$

to commute for any path $p : x \to x'$ in $X$. But this commutativity follows from looking at the composite map

$$\Delta_1 \times \Delta_1 \xrightarrow{p \times \mathrm{id}} X \times \Delta_1 \xrightarrow{H} Y.$$

**Remark 6.6.15.** In particular, if $f : X \to Y$ is a homotopy equivalence of anima, then $\pi_{\leq 1}(f)$ is an equivalence of categories.

### 6.6.3 Higher morphisms

For higher morphisms, we can define composition inductively.

**Definition 6.6.16.** Let $X$ be an $\infty$-groupoid, and $f, g, h$ be three parallel left $n$-morphisms, with source and target $s, t$.

Suppose $\eta \in \mathrm{Mor}_X^{(n+1),L}(f, g)$ and $\eta' \in \mathrm{Mor}_X^{(n+1),L}(g, h)$. We say that $\eta'' \in \mathrm{Mor}_X^{(n+1),L}(f, h)$ is a *composition* of $\eta', \eta$ if, under the identification of Construction 6.5.13, $\eta''$ is a composition of the 1-morphisms $\eta', \eta$ of the $\infty$-groupoid $\mathrm{Mor}_X^{n,L}(s, t)$.

# Chapter 7

# The long exact sequence of a fibration

## 7.1 Introduction

Let $f : X \to S$ be a functor between $\infty$-groupoids. Suppose $s \in S_0$ is a basepoint, and $F$ is the homotopy fiber of $f$ above $s$.

The functor $f$ is some 'animation' of the function

$$\pi_0(f) : \pi_0(X) \to \pi_0(S)$$

of sets. How, then, does $\pi_0(F)$ relate to the fiber of $\pi_0(f)$ above $[s]$?

The projection $F \to X$ induces a map $\pi_0(F) \to \pi_0(X)$. It's easy to check that the image of this map is contained in the fiber of $\pi_0(f)$ above $[s]$; indeed, if $(x, e) \in F_0$, then $e$ is a path from $f(x)$ to $s$, and so of course $[f(x)] = [s]$. Thus we have a natural map

$$\pi_0(F) \to \pi_0(f)^{-1}([s]).$$

However, this map $\pi_0(F) \to \pi_0(f)^{-1}([s])$ is not always injective, because a single point $x$ might have $[f(x)] = [s]$ for *multiple reasons*.

Thus $\pi_0(F)$ can be much larger than the fiber of $\pi_0(f)$.

**Ideation 7.1.1.** Can we at least describe how non-injective $\pi_0(F) \to \pi_0(X)$ is?

Take $x \in X_0$ some 0-simplex so that the fiber of $\pi_0(F) \to \pi_0(X)$ above $[x]$ is nonempty. Every path from $f(x)$ to $s$ gives a reason for $x$ to lie in the homotopy fiber. As homotopy classes of paths from $f(x)$ to $s$ form a $\pi_1(S, s)$-torsor, we might expect that the fiber of $\pi_0(F) \to \pi_0(X)$ above $[x]$ is a copy of $\pi_1(S, s)$.

And indeed, we can give a map

$$\delta : \pi_1(S, s) \to \pi_0(F),$$

$$\delta([\ell]) := [(x, \ell \circ e)],$$

where $e \in S_1$ is some arbitrarily chosen path from $f(x)$ to $s$.

And while $\delta$ *is* surjective onto the fiber of

$$\pi_0(F) \to \pi_0(X)$$

above $[x]$, $\delta$ *is again not injective!*

To see why, observe that a path from $(x, e)$ to $(x, \ell \circ e)$ in $F$ is the data of a loop $\ell' \in X_1$ from $x$ to itself, together with a map

$$\Delta_1 \times \Delta_1 \to S$$

having boundary

$$\begin{array}{ccc} f(x) & \xrightarrow{\ell \circ e} & s \\ {\scriptstyle f(\ell')}\uparrow & & \uparrow{\scriptstyle s} \\ f(x) & \xrightarrow{e} & s. \end{array}$$

In particular, whenever we can find a loop $\ell'$ so that $[\ell] = [e] \circ [f(\ell')]^{-1} \circ [e]^{-1}$ (in $\pi_1(S, s)$), we will have $\delta([\ell]) = \delta(1)$.

This suggests that the failure of injectivity of

$$\pi_1(S, s) \xrightarrow{\delta} \pi_0(F)$$

is controlled by $\pi_1(X, x)$.

Putting this all together, we get the *monodromy sequence*.

**Notation 7.1.2.** Let $(X, x)$ be a pointed simplicial set. We write $\pi_0(X, x)$ for the *pointed* set $\pi_0(X)$, with basepoint $[x]$.

**Theorem 7.1.3** (Monodromy sequence)**.** *Let $f : (X, x) \to (S, s)$ be a morphism of anima, with homotopy fiber $F$. Then there is an exact sequence of* pointed sets

$$\pi_1(X, x) \xrightarrow{\pi_1(f)} \pi_1(S, s) \xrightarrow{\delta} \pi_0(F, (x, c_s)) \to \pi_0(X, x) \to \pi_0(S, s),$$

*where*

$$\delta : \pi_1(S, s) \to \pi_0(F, (x, c_s))$$

*is a function which can be constructed functorially in $f$.*

*Proof.* First we check exactness of

$$\pi_0(F, (x, c_s)) \to \pi_0(X, x) \to \pi_0(S, s).$$

The composite is constant with value $[s]$. If $[x'] \in \pi_0(X, x)$ has $[f(x')] = [s]$, then there is a path $e \in S_1$ from $f(x')$ to $s$, and so $(x', e) \in F_0$ proves that the image of $\pi_0(F, (x, c_s)) \to \pi_0(X, x)$ contains $[x']$. This shows exactness at $\pi_0(X, x)$.

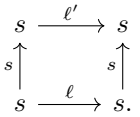
To construct $\delta$, let

$$\delta([\ell]) := [(x, \ell)].$$

To see this is well-defined, we first show that if $\ell \simeq \ell'$ in $\pi_1(S, s)$, then $(x, \ell)$ and $(x, \ell')$ lie in the same path component of $F$. And indeed, the homotopy in $\pi_1(S, s)$ tells us we can find a map

$$\Delta_1 \times \Delta_1 \to S$$

with boundary



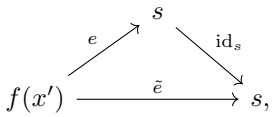

Thus the pair of this square, and the constant path $x : \Delta_1 \to X$, give a path from $(x, \ell)$ to $(x, \ell')$ in the homotopy fiber, showing $\delta$ is well-defined.

Now, we show that

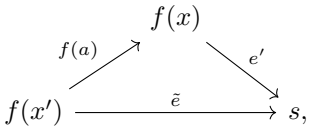
$$\pi_1(S, s) \xrightarrow{\delta} \pi_0(F, (x, c_s)) \to \pi_0(X, x)$$

is exact. Again, it is clear that the composite map is the constant $[x]$. Take now some $(x', e) \in F_0$ which maps to $[x]$. Thus $[x'] = [x]$, and so we may find an edge $a \in X_1$ from $x'$ to $x$.

We claim first that there's some $e' \in S_1$ so that $[(x', e)] = [(x, e')]$. And indeed, this can be done with the following horn fillings. First, fill a $\Lambda^1_2$ horn to get a 2-simplex




where $\tilde{e}$ is just our name for the new edge. Filling a $\Lambda^0_2$-horn, we get another 2-simplex




This produces our $e' \in S_1$. And glueing these two 2-simplices together alongside $\tilde{e}$, we get a map $\Delta_1 \times \Delta_1 \to S$, which curries to a path from $(x', e)$ to $(x, e')$ in $F$.

This $e'$ is then an edge in $S$ from $f(x) = s$ to $s$, and so

$$\delta([e']) = [(x, e')],$$

proving exactness at $\pi_0(F)$.

Finally, we prove that

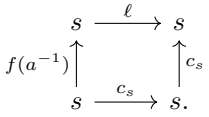
$$\pi_1(X, x) \to \pi_1(S, s) \xrightarrow{\delta} \pi_0(F, (x, c_s))$$

is exact. The composite is constant by the argument of Ideation 7.1.1.

Take some $[\ell] \in \pi_1(S, s)$ with $\delta([\ell]) = [(x, c_s)]$. Thus $[(x, \ell)] = [(x, c_s)]$, meaning there exists some loop $[a] \in \pi_1(X, x)$ such that we have a map

$$\Delta_1 \times \Delta_1 \to S$$

with boundary




Then $\pi_1(f)([a]) = [f(a)]$. But the above square proves $[f(a)] = [\ell]$, and so we have our desired exactness. □

## 7.2 The monodromy sequence in the case of a Kan fibration

When $f : X \to S$ is a *Kan fibration* of anima, because the homotopy fiber and the strict fiber are equivalent, we can use $X_s$ in place of $F$ in the monodromy sequence.

It is helpful in this case to spell out what the boundary map $\delta$ is, in terms of $X_s$.

**Construction 7.2.1.** We construct a map

$$\delta : \pi_1(S, s) \to \pi_0(X_s, x).$$

Take a homotopy class $[\ell] \in \pi_1(S, s)$. Lifting along the anodyne morphism $\{0\} \to \Delta_1$, we can find some edge $e$ of $X$ such that $f(e) = \ell$, and $d^1(e) = x$. Define $\delta([\ell]) = [d^0(e)]$. Note that $e$ is an edge of $X$, but not necessarily an edge of $X_s$, and so $\delta$ can be (and often is!) non-constant.

This construction of $\delta$ is independent of the choice of representative of $[\ell]$ by an easy horn filling argument. For independence of the choice of lift, note that Theorem 4.8.9 implies any two choices of $e$ are homotopic via a homotopy which is constant on the left endpoint $x$, and which projects down to the constant homotopy from $\ell$ to itself. The right endpoint of this homotopy then witnesses a path in $X_s$ from the right endpoints of the two choices of $e$, proving they are equal in $\pi_0(X_s, x)$.

**Proposition 7.2.2.** *Let $f : (X, x) \to (S, s)$ be a Kan fibration of anima. Let $i : X_s \hookrightarrow F$ denote the canonical inclusion.*

*Then, up to composition with $\pi_0(i)$ to identify the targets, the boundary map $\delta$ of Construction 7.2.1 and the boundary map $\delta'$ constructed in the proof of Theorem 7.1.3 are equivalent: $\pi_0(i) \circ \delta = \delta'$.*

*Proof.* For any $[\ell] \in \pi_1(S, s)$, by definition we have $\delta'([\ell]) := [(x, \ell)]$.

So, it suffices to prove that, if $\tilde{\ell}$ is any lift of $\ell$ to a path $\tilde{\ell} : x \to x'$, then $(x', c_s)$ and $(x, \ell)$ lie in the same path component of the homotopy fiber $F$. But $\tilde{\ell}$ itself witnesses the path between these two points, and so we conclude. □

## 7.3 Application: The fundamental group of a circle

Armed with the monodromy sequence, we can calculate an interesting example of a fundamental group!

### 7.3.1 Simplicial groups

Unfortunately for us, we cannot use the monodromy sequence to compute $\pi_1(\mathrm{Sing}(S^1))$ as a group, since the $\pi_0$'s involved are sets, and so the monodromy sequence is only an exact sequence of pointed sets. To get around this, we now develop a bit of the theory of simplicial groups.

We start with a fun bit of pure algebra.

**Theorem 7.3.1** (Eckmann–Hilton)**.** *Let $A$ be a set with two binary operations $\circ$ and $\otimes$, such that*

1. *both operations are unital, with possibly different units $1_\circ, 1_\otimes$,*
2. *we have $(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ for all $a, b, c, d \in A$.*

*Then $\circ = \otimes$ and this operation is associative and commutative.*

*Proof.* Note that

$$\begin{aligned} 1_\circ &= 1_\circ \circ 1_\circ \\ &= (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) \\ &= (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) \\ &= 1_\otimes \otimes 1_\otimes \\ &= 1_\otimes, \end{aligned}$$

so the two units coincide. Then

$$(a \otimes 1) \circ (1 \otimes b) = (a \circ 1) \otimes (1 \circ b),$$

so that $a \circ b = a \otimes b$, and thus our two operations coincide.

Associativity is now because

$$\begin{aligned} (a \otimes b) \otimes c &= (a \otimes b) \otimes (1 \otimes c) \\ &= (a \otimes 1) \otimes (b \otimes c) \\ &= a \otimes (b \otimes c), \end{aligned}$$

and commutativity is because

$$\begin{aligned} a \otimes b &= (1 \otimes a) \otimes (b \otimes 1) \\ &= (1 \otimes b) \otimes (a \otimes 1) \\ &= b \otimes a. \end{aligned}$$

□

**Definition 7.3.2.** A *simplicial group* is a group object in the category of simplicial sets.

Eckmann–Hilton has a surprising number of applications! As an example, suppose we have a simplicial group $G$ which is also a Kan complex. Then there are two natural ways to compose loops: the usual composition operation, *or* compose them by pointwise multiplication. Thanks to Eckmann–Hilton, these two operations coincide and are abelian!

**Proposition 7.3.3.** *Let $G$ be a simplicial group which is also a Kan complex. Let $e \in G$ be its identity. Given $\ell, \ell' \in [\Delta_1/\partial\Delta_1, (G, e)]_*$, define*

$$\ell\ell' : \Delta_1 \to G$$

*the composite map*

$$\Delta_1 \xrightarrow{(\ell,\ell')} G \times G \xrightarrow{\mu} G.$$

*Then $[\ell][\ell'] = [\ell\ell']$ in $\pi_1(G, e)$. Moreover, $\pi_1(G, e)$ is abelian.*

**Warning 7.3.4.** The group $G$ can be highly non-abelian in general! It is only the operation on $\pi_1(G, e)$ that is forced to be abelian.

*Proof.* We have two different group structures on the set $\pi_1(G, e)$, namely the usual one of concatenation, and this new operation of multiplication. We denote the usual one by $\circ$, and this new one by $*$. Note that they are both unital, in fact even with the same unit.

We claim now that

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d).$$

Indeed, this is just because

$$\pi_1(G \times G, (e, e)) = \pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G, e)$$

is a group homomorphism. By Eckmann-Hilton, we deduce that $* = \circ$. □

**Remark 7.3.5.** When $G$ is a simplicial group, the pointed set $\pi_0(G, e)$ is actually also a group: indeed, product-preserving functors always send group objects to group objects.

**Proposition 7.3.6.** *Let $f : X \to S$ be a Kan fibration of simplicial groups, such that $f$ is also a homomorphism of group objects and $X, S$ are anima. Set $x = 1_X, s = 1_S$. Then the monodromy sequence*

$$\pi_1(X, x) \to \pi_1(S, s) \xrightarrow{\delta} \pi_0(X_s, x) \to \pi_0(X, x) \to \pi_0(S, s)$$

*is an exact sequence of groups.*

*Proof.* The only thing to check is that $\delta$ is a group homomorphism. Take $[\ell], [\ell'] \in \pi_1(S, s)$. We will compute $\delta$ using its description from Construction 7.2.1.

Choose $e, e' \in X_1$ two edges lifting $\ell, \ell'$, both having left endpoint $x$. Then $\delta([\ell]) * \delta([\ell']) = [d^0(e)] * [d^0(e')]$, with $*$ the group operation on $X$ (which descends to a group operation on $X_s$).

As $f$ is a group homomorphism, we have $f(e * e') = \ell * \ell'$, and hence

$$\delta([\ell * \ell']) = [d^0(e * e')] = [d^0(e)] * [d^0(e')].$$

This implies $\delta$ is a group homomorphism, so we conclude. □

### 7.3.2 The fundamental group of the circle

**Theorem 7.3.7.** *Set $S = \mathrm{Sing}(S^1)$. Then $\pi_1(S, 1) \simeq (\mathbb{Z}, +)$.*

*Proof.* We first claim that the map

$$f : \mathrm{Sing}(\mathbb{R}) \to \mathrm{Sing}(S^1)$$

obtained by applying Sing to the natural surjection $\mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ is a Kan fibration; indeed, this follows immediately from Theorem 5.3.9, as $\mathbb{R} \to S^1$ is a locally trivial fibration with fiber $\mathbb{Z}$. Note that $f$ is also a homomorphism of simplicial groups (as product-preserving functors send group objects to group objects).

With Theorem 4.5.16, one can easily check $\mathrm{Sing}(\mathbb{R}) \simeq \mathrm{pt}$. Thus the monodromy sequence

$$\pi_1(\mathbb{R}, 0) \to \pi_1(S^1, 1) \to \pi_0(\mathbb{Z}, 0) \to \pi_0(\mathbb{R}, 0) \to \pi_0(S^1, 1)$$

proves that there is a bijection *of sets* $\pi_1(S^1, 1) \simeq \mathbb{Z}$. But by Proposition 7.3.6, this is actually a group isomorphism $\pi_1(S^1, 1) \simeq (\mathbb{Z}, +)$, so we conclude. □

**Remark 7.3.8.** A locally trivial fibration with discrete fiber is called a *covering space*. As we just saw, covering spaces are incredibly convenient to compute with – one common technique for computing fundamental groups is to try and find a contractible covering space.

## 7.4 Application: Path lifting with endpoints

For any Kan fibration $f : X \to S$, if we're given a path $p$ in $S$ and a lift of $p(0)$ to $X$, then we can lift the entire path $p$ starting at our lift of $p(0)$. In some situations, it is of interest to know if we can find a lift of $p$ with *both* endpoints fixed. Fortunately, the monodromy sequence allows us to sometimes accomplish this stronger form of path lifting.

**Theorem 7.4.1** (Path lifting with endpoints)**.** *Let $f : X \to S$ be a Kan fibration of Kan complexes. Suppose that $x_0, x_1 \in X_0$ are two points, and $p : f(x_0) \to f(x_1)$ is a path between them.*

*Assume that*

$$\pi_0(X) \to \pi_0(S)$$

*is* injective *and that*

$$\pi_1(X, x_1) \to \pi_1(S, f(x_1))$$

*is* surjective*.*

*Then, there exists a path $\tilde{p} : x_0 \to x_1$ in $X$ such that $f(\tilde{p}) = p$.*

*Proof.* Let $F$ be the homotopy fiber of $f$ above $f(x_1)$. The monodromy sequence

$$\pi_1(X, x_1) \to \pi_1(S, f(x_1)) \to \pi_0(F) \to \pi_0(X, x_1) \to \pi_0(S, f(x_1))$$

implies $\pi_0(F) = \mathrm{pt}$, and so $F$ is connected, as desired.

In particular, there is a path in $F$ from $(x_0, p)$ to $(x_1, c_{f(x_1)})$. This consists of a path $q : x_0 \to x_1$ together with a map $\Delta_1 \times \Delta_1 \to S$ having boundary

$$\begin{array}{ccc} f(x_1) & \xrightarrow{c_{f(x_1)}} & f(x_1) \\ {\scriptstyle f(q)}\uparrow & & \uparrow{\scriptstyle f(x_1)} \\ f(x_0) & \xrightarrow{p} & f(x_1). \end{array}$$

This square proves $f(q) \simeq p$ rel endpoints.

Consider now the map

$$\theta : [\Delta_1, X] \to [\{0,1\}, X] \times_{[\{0,1\}, S]} [\Delta_1, S].$$

The point $q \in [\Delta_1, X]$ together with the homotopy $f(q) \simeq p$ gives us a point of the homotopy fiber of $\theta$ above $((x_0, x_1), p)$. As $\theta$ is a Kan fibration, it follows that the strict fiber above $((x_0, x_1), p)$ is non-empty as well; any vertex of this strict fiber is a lift of $p$ with the desired properties. □

As a corollary, we can prove that a Kan fibration is a homotopy equivalence if and only if it is a trivial Kan fibration.

**Corollary 7.4.2.** *Let $f : X \to S$ be a Kan fibration of anima. Assume that $f$ is a homotopy equivalence. Then $f$ is a trivial Kan fibration.*

*Proof.* By Corollary 5.4.5, it suffices to check that every lifting problem

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\bar{b}} & S \end{array}$$

with $i : A \hookrightarrow B$ an inclusion has a solution up to homotopy. When $A = \emptyset$, this is actually quite easy: choose $g : S \to X$ a homotopy inverse of $f$, and then observe that $g \circ \bar{b}$ is a solution up to homotopy. In particular, by applying this case to $\emptyset \hookrightarrow S$, we can deduce that $f : X \to S$ admits a section $s : S \to X$. As $f$ is invertible up to homotopy, this $s$ must be a homotopy inverse to $f$.

The map

$$f_* : [X, X] \to [X, S]$$

defined by postcomposition with $f$ is a Kan fibration by Theorem 4.8.13. It is also a homotopy equivalence. In fact, we claim

$$s_* : [X, S] \to [X, X],$$

is a homotopy inverse.

Observe first that $f_* s_* = (fs)_* = \mathrm{id}_* = \mathrm{id}$.

For the composition $s_* f_*$, choose a homotopy $H : \Delta_1 \times X \to X$ from $sf$ to $\mathrm{id}_X$. Define

$$\tilde{H} : \Delta_1 \times [X, X] \to [X, X]$$

the map adjunct to

$$\Delta_1 \times [X, X] \times X = \Delta_1 \times (X \times [X, X]) \xrightarrow{\mathrm{id} \times \mathrm{eval}_X} \Delta_1 \times X \xrightarrow{H} X.$$

At time 0, $\tilde{H}$ is adjunct to

$$X \times [X, X] \xrightarrow{\mathrm{eval}_X} X \xrightarrow{sf} X,$$

which of course is just $(sf)_* = s_* f_*$. At time 1, we similarly see that $\tilde{H}$ is the identity.

Thus $f_*$ is a homotopy equivalence, and in particular induces an equivalence on $\pi_{\leq 1}$. Applying Theorem 7.4.1 to $f_*$, we deduce that we can solve all lifting problems of $\{0, 1\} \hookrightarrow \Delta_1$ against $f_*$. In particular, we may choose some homotopy

$$H : \Delta_1 \times X \to X$$

from $sf$ to $\mathrm{id}_X$ such that $f \circ H$ is the constant homotopy from $fsf = f$ to itself.

Set $b' := s \circ \bar{b}$. Then $f \circ b' = \bar{b}$, and $b'|_A = s\bar{b}i = sfa \simeq a$. In particular, $f \circ (H \circ a)$ is the constant homotopy from $fb'$ to $\bar{b}$, and our original lifting problem has a solution up to homotopy, as desired. □

## 7.5 Continuing the monodromy sequence

Let $f : (X, x) \to (S, s)$ be a Kan fibration of pointed Kan complexes. Can we extend the monodromy sequence

$$\pi_1(X, x) \to \pi_1(S, s) \xrightarrow{\delta} \pi_0(X_s, x) \to \pi_0(X, x) \to \pi_0(S, s)$$

further to the left?

**Remark 7.5.1.** The monodromy sequence is a bit more concrete for Kan fibrations, as it uses strict fibers instead of homotopy fibers, which is why we start with this case; but of course there is a parallel theory for arbitrary morphisms of anima, and we will develop it soon.

Recall that $\pi_1(X,x) = \pi_0\Omega(X,x)$. For convenience, we'll omit the basepoints in our loop spaces and just write $\Omega X$ in this section; $x$ will always be the basepoint of $X$ and $X_s$, and $s$ will always be the basepoint of $S$.

Anyways, we can rewrite the monodromy sequence as an exact sequence

$$\pi_0(\Omega X) \to \pi_0(\Omega S) \xrightarrow{\delta} \pi_0(X_s) \to \pi_0(X) \to \pi_0(S).$$

From this perspective, there's an obvious guess as to how to extend the monodromy sequence: study the map $\Omega X \to \Omega S$, and hope it's a Kan fibration.

**Lemma 7.5.2.** *If $f : X \to S$ is a Kan fibration of pointed Kan complexes, then*

$$\Omega(f) : \Omega(X) \to \Omega(S)$$

*is a Kan fibration, and the fiber of $\Omega(f)$ is isomorphic to the loop space of the fiber of $f$.*

*Proof.* To see that $\Omega(f)$ is a Kan fibration, suppose we have a commutative diagram

$$\begin{array}{ccc} \Lambda^i_n & \longrightarrow & \Omega(X) \\ \downarrow & & \downarrow{\scriptstyle \Omega(f)} \\ \Delta_n & \xrightarrow{\sigma} & \Omega(S). \end{array}$$

By the tensor-hom adjunction, a map $\Delta_n \to \Omega(X)$ is the same thing as a map $\Delta_n \times \Delta_1 \to X$ whose restriction to $\Delta_n \times \{0,1\}$ is the constant map with value $x$, and a similar description holds for maps $\Lambda^i_n \to \Omega(X)$.

By Theorem 4.8.8, using that horn inclusions are strongly anodyne, we see that

$$(\Delta_n \times \{0,1\}) \coprod_{\Lambda^i_n \times \{0,1\}} (\Lambda^i_n \times \Delta_1) \hookrightarrow \Delta_n \times \Delta_1$$

is anodyne; but then our lifting problem for $\Omega(f)$ can be rephrased as a lifting problem for $f$ against this anodyne map, so we can solve it.

Next, we must check that the fiber of $\Omega(f)$ is $\Omega(X_s)$. And indeed, the fiber of $\Omega(f)$ is

$$\Omega(X) \times_{\Omega(S)} \{c_s\},$$

where $c_s \in \Omega(S)_0$ denotes the constant loop at $s$.

This simplicial set has the universal property that a map

$$A \to \Omega(X) \times_{\Omega(S)} \{c_s\}$$

is equivalent to a map

$$\varphi : A \to \Omega(X)$$

such that $\Omega(f) \circ \varphi$ is the constant map with value $c_s$. Maps $\varphi : A \to \Omega(X)$ are equivalent to maps

$$\varphi' : A \times \Delta_1 \to X$$

such that $\varphi'|_{A \times \{0,1\}}$ is the constant map with value $x$.

Saying that the corresponding map $\varphi$ has $\Omega(f) \circ \varphi$ constant is equivalent to saying that $f \circ \varphi'$ is constant with value $s$. But this is equivalent to saying that $\varphi'$ factors through $X_s \hookrightarrow X$, and hence $\varphi'$ curries to a map $A \to \Omega(X_s)$. Thus $\Omega(X_s)$ has the same universal property as the fiber of $\Omega(f)$, so we conclude. □

The monodromy sequence can therefore be applied to $\Omega(f)$; doing this inductively, we see that the monodromy sequence extends to a long exact sequence of pointed sets

$$\cdots \to \pi_0(\Omega^n X_s) \to \pi_0(\Omega^n X) \to \pi_0(\Omega^n S) \xrightarrow{\delta_n} \pi_0(\Omega^{n-1} X_s) \to \cdots . \tag{7.5.3}$$

We will eventually dub (7.5.3) the *long exact sequence of a fibration*. At the moment, though, this long exact sequence is quite inexplicit; in the next section, we will try to make it more concrete, by

1. describing $\pi_0(\Omega^n X)$ and its group structure more directly in terms of $X$,
2. proving that the boundary maps $\delta_n$ (for $n \geq 2$) are group homomorphisms.

After doing that, we will give (7.5.3) its rightful name.

## 7.6 The higher homotopy groups

**Definition 7.6.1.** Let $(X, x)$ be a pointed Kan complex, and take $n \geq 2$. We define the $n^{th}$ *higher homotopy group of* $(X, x)$ to be

$$\pi_n(X, x) := \pi_1(\Omega^{n-1}(X, x)).$$

**Remark 7.6.2.** Note that $\pi_n(X, x) = \pi_0(\Omega^n(X, x))$, and in particular these are exactly the groups which appear in the long exact sequence of a fibration.

**Construction 7.6.3** (Functoriality)**.** Suppose $n \geq 2$. Let $f : (X, x) \to (Y, y)$ be a morphism of pointed Kan complexes. We define a group homomorphism

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$$

as the group homomorphism

$$\pi_n(X, x) = \pi_1(\Omega^{n-1}(X, x)) \to \pi_1(\Omega^{n-1}(Y, y)) \to \pi_n(Y, y),$$

where the middle map is the one induced by the functor $\pi_{\leq 1}(\Omega^{n-1}(f))$.

**Remark 7.6.4.** If there is a *pointed homotopy* from $f : (X, x) \to (Y, y)$ to $g : (X, x) \to (Y, y)$ (where pointed homotopy means the homotopy always sends $x$ to $y$), then $\pi_n(f) = \pi_n(g)$. This is because a pointed homotopy from $f$ to $g$ is easily upgraded to a homotopy from $\Omega^{n-1}(f)$ to $\Omega^{n-1}(g)$. But if our maps are not pointed homotopic, this argument fails.

**Proposition 7.6.5.** *Let $(X, x)$ be a pointed Kan complex. For every $i \geq 1, j \geq 0$, there is a (functorial in $(X, x)$) isomorphism of groups*

$$\pi_{i+j}(X, x) = \pi_i(\Omega^j(X, x)).$$

*Proof.* Both sides are defined to be $\pi_1(\Omega^{i+j-1}(X, x))$. □

These iterated loop spaces are special cases of morphism spaces in an $\infty$-groupoid. Thus our study of left morphisms in the last chapter gives a simplicial model for these higher homotopy groups (at the moment, naively uncurrying lets us treat $\pi_n(X, x)$ as the set of homotopy classes of maps $\Delta^n_1 \to X$ *rel* the boundary, and this is quite an awkward set to deal with, as the cuboid $\Delta^n_1$ is annoying simplicially).

**Theorem 7.6.6.** *For every $n \geq 0$ and pointed Kan complex $(X, x)$, we have a homotopy equivalence of anima*

$$\Omega^n(X, x) \simeq [\Delta_n/\partial\Delta_n, (X, x)]_*.$$

*In fact, this is even a homotopy equivalence of pointed anima (with constant maps as basepoints on both sides).*

**Remark 7.6.7.** When $n = 0$, we have

$$\Delta_0/\partial\Delta_0 = \Delta_0 \coprod_{\emptyset} \mathrm{pt},$$

so that it is a disjoint union of two points, exactly what we'd expect the 0-sphere to be.

*Proof.* Let $e_k \in X_k$ denote the totally degenerate $k$-simplex at $x$ (that is, the image of $x$ under the unique degeneracy map $X_0 \to X_k$). Then $e_k$ is a left $k$-morphism with source and target $e_{k-1}$. By Theorem 6.5.6,

$$\mathrm{Mor}_X^{(n+1),L}(e_n, e_n) \simeq \mathrm{Mor}_{\mathrm{Mor}_X^{n,L}(e_{n-1},e_{n-1})}(e_n, e_n) = \Omega(\mathrm{Mor}_X^{n,L}(e_{n-1}, e_{n-1}), e_n).$$

As this homotopy equivalence can be chosen to come from a restriction map (see Construction 6.5.13), we can choose it to send $e_{n+1}$ to $\mathrm{id}_{e_n}$.

Unwinding definitions, we have an isomorphism of simplicial sets

$$\mathrm{Mor}_X^{(n+1),L}(e_n, e_n) \simeq [\Delta_{n+1}/\partial\Delta_{n+1}, (X, x)]_*,$$

with $e_{n+1}$ corresponding to the constant map with value $x$.

Our claim therefore follows immediately from induction on $n$. □

**Corollary 7.6.8.** *For every $n \geq 1$, we have an isomorphism of pointed sets*

$$\pi_n(X, x) \simeq \pi_0[\Delta_n/\partial\Delta_n, (X, x)]_*.$$

**Remark 7.6.9.** It is possible to describe the group structure on $\pi_n$ purely in terms of maps $\Delta_n/\partial\Delta_n \to (X, x)$, but we will not do this in these notes.

The results of Section 6.3 on morphism spaces can be applied to higher homotopy groups, giving us some results on independence of basepoints and homotopy invariance.

**Construction 7.6.10** (Independence of basepoint)**.** Let $X$ be a Kan complex, and suppose $x, x' \in X_0$ lie in the same path component of $X$. We construct, for any path $p : x \to x'$, an isomorphism of groups

$$\psi_p : \pi_n(X, x) \simeq \pi_n(X, x')$$

for any $n \geq 2$.

Define $\gamma_p : \Delta_1 \to [\Delta_1, X]$ the 1-simplex corresponding to

$$\Delta_1 \times \Delta_1 \to X, (t, s) \mapsto p(t).$$

In particular,

$$\gamma_p(t) = c_{p(t)}$$

is the constant path.

Consider the commutative diagram

$$\begin{array}{ccc} (\Omega(X,x)\times\{0\})\cup(\{c_x\}\times\Delta_1) & \longrightarrow & [\Delta_1, X] \\ \downarrow & \overset{\psi}{\nearrow} & \downarrow \\ \Omega(X,x)\times\Delta_1 & \xrightarrow{(p,p)\circ\pi_2} & X\times X. \end{array}$$

Here, the top map is inclusion on $\Omega(X,x)\times\{0\}$ and is $\gamma_p$ on $\{c_x\}\times\Delta_1$.

As the right map is a Kan fibration and the left map is anodyne, we can find a map $\psi$ solving the resulting lifting problem; this $\psi$ will be unique up to homotopy. Then

$$\psi(\ell, 0) = \ell,$$

and $\psi(\ell,1) \in \Omega(X,x')$ (by compatibility with the lower square). Thus $\psi$ restricted to $\Omega(X,x)\times\{1\}$ gives a map

$$T_p : \Omega(X,x) \to \Omega(X,x').$$

This map $T_p$ is a homotopy equivalence (by Theorem 5.3.2), and – using that the restriction of $\psi$ to $\{c_x\}\times\Delta_1$ is $\gamma_p$ – has

$$T_p(c_x) = (c_{x'}).$$

As $\psi$ is well-defined up to homotopy rel $\{c_x\}\times\Delta_1$, this map $T_p$ is well-defined up to homotopy rel $c_x$.

Thus $T_p$ induces a group isomorphism

$$\psi_p = \pi_{n-1}(T_p) : \pi_n(X,x) \to \pi_n(X,x'),$$

which is independent of our choice of $T_p$ by Remark 7.6.4.

**Remark 7.6.11.** The independence when $n = 1$ is immediate from the homotopy invariance of the fundamental groupoid.

**Proposition 7.6.12** (Homotopy invariance of higher homotopy groups)**.** *Let $f, g : X \to Y$ be two homotopic maps of Kan complexes. Choose*

$$H : X\times\Delta_1 \to Y$$

*a homotopy between them. Fix $x \in X$ a basepoint, and let $H_x : \{x\}\times\Delta_1 \to Y$ be the resulting path; call its endpoints $y, y'$.*

*Then the diagram*

$$\begin{array}{ccc} \pi_n(X,x) & \xrightarrow{\pi_n(f)} & \pi_n(Y,y) \\ & \searrow^{\pi_n(g)} & \downarrow{\scriptstyle \psi_{H_x}} \\ & & \pi_n(Y,y') \end{array}$$

*commutes for all $n \geq 2$, where $\psi_{H_x}$ is the map of Construction 7.6.10.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} \Delta_1\times[\Delta_1, X] & \xrightarrow{\tilde{H}} & [\Delta_1, Y] \\ \downarrow & & \downarrow \\ \Delta_1\times X\times X & \xrightarrow{H'} & Y\times Y, \end{array}$$

where horizontal maps are the obvious restrictions and

$$\tilde{H}(t,p)(s) = H(p(s),t),$$

$$H'(t,x_1,x_2) = (H(x_1,t), H(x_2,t)).$$

By a small modification of the argument in Theorem 5.3.4, the diagram

$$\begin{array}{ccc} \Omega(X,x) & \xrightarrow{\Omega(f)} & \Omega(Y,y) \\ & \searrow^{\Omega(g)} & \downarrow{\scriptstyle T_{H_x}} \\ & & \Omega(Y,y') \end{array}$$

commutes up to basepoint preserving homotopy. Taking $\pi_{n-1}$, we conclude. □

Remarkably, the higher homotopy groups are all abelian.

**Theorem 7.6.13.** *For every $n \geq 2$, and every pointed Kan complex $(X,x)$, the group $\pi_n(X,x)$ is abelian.*

*Proof.* By definition,

$$\pi_n(X,x) = \pi_1(\Omega^{n-1}X).$$

For $n \geq 2$, a loop in $\Omega^{n-1}X$ is the same as a map

$$\Delta_1 \times \Delta_1 \to \Omega^{n-2}X$$

sending the outer four edges to $x$. Let $O_2 \subseteq \Delta_1 \times \Delta_1 =: Q_2$ denote the subset consisting of those outer four edges.

Thus

$$\pi_n(X,x) = \pi_0[Q_2/O_2, \Omega^{n-2}(X,x)]_*.$$

There are *two* natural group operations on the set $\pi_0[Q_2/O_2, \Omega^{n-2}(X,x)]_*$, though: one could compose maps by using the first $\Delta_1$ factor, *or* using the second. The first of these operations is just the group operation on $\pi_n(X,x)$, and the second is some other operation.

By the Eckmann–Hilton argument (see Theorem 7.3.1), to prove $\pi_n(X,x)$ is abelian it suffices to show that these two operations distribute over each other. Denote the first (the usual group operation) by $\otimes$, and the second (this new operation) by $\circ$. More precisely, we hope to prove

$$([a] \otimes [b]) \circ ([c] \otimes [d]) = ([a] \circ [c]) \otimes ([b] \circ [d]).$$

Let's begin. To find $[a] \otimes [b]$ and $[c] \otimes [d]$, we take the map

$$ab, cd : \Lambda^1_2 \times \Delta_1 \to \Omega^{n-2}(X)$$

whose 01 edge is $a$ (resp. $c$) and whose 12 edge is $b$ (resp. $d$), and then we fill to maps

$$ab, cd : \Delta_2 \times \Delta_1 \to \Omega^{n-2}X.$$

By lifting along the anodyne map

$$(\Lambda^1_2 \times \Delta_1) \cup (\Delta_2 \times \partial\Delta_1) \hookrightarrow \Delta_2 \times \Delta_1,$$

we can assume these fillings send $\Delta_2 \times \partial\Delta_1$ to the basepoint $x$.

The 02 edges are representatives of $a \otimes b$, resp. $c \otimes d$. Again, combine these into maps $\Delta_2 \times \Lambda^1_2 \to \Omega^{n-2}X$ and extend along the anodyne inclusion

$$(\Delta_2 \times \Lambda^1_2) \coprod_{\{0,1,2\}\times\Lambda^1_2} (\{0,1,2\} \times \Delta_2) \hookrightarrow \Delta_2 \times \Delta_2$$

to get a map

$$H : \Delta_2 \times \Delta_2 \to \Omega^{n-2}(X)$$

whose restriction to $02 \times 02$ is a representative of $([a] \otimes [b]) \circ ([c] \otimes [d])$.

But take this same map $H$, and think of its restriction to $\Delta_2 \times 01$. This is just $ab$. Similarly, its restriction to $\Delta_2 \times 12$ is just $cd$. Thus

1. the restriction of $H$ to $01 \times 01$ is $a$,
2. the restriction of $H$ to $01 \times 12$ is $c$,
3. and hence the restriction of $H$ to $01 \times 02$ is a representative of $[a] \circ [c]$.

Likewise, the restriction of $H$ to $12 \times 02$ is a representative of $[b] \circ [d]$. Hence its restriction to $02 \times 02$ must be a representative of

$$([a] \circ [c]) \otimes ([b] \circ [d]),$$

and so we conclude. □

## 7.7 The long exact sequence of the homotopy fiber

**Theorem 7.7.1** (The long exact sequence of the homotopy fiber)**.** *Let $f : (X, x) \to (S, s)$ be a morphism of pointed anima. Let $(F, x)$ denote the homotopy fiber. Then there exists a long exact sequence of pointed sets*

$$\cdots \to \pi_n(F, x) \to \pi_n(X, x) \to \pi_n(S, s) \xrightarrow{\delta_n} \pi_{n-1}(F, x) \to \cdots,$$

*functorial in $f$, such that $\delta_n$ is a group homomorphism for all $n \geq 2$.*

*Moreover, if $f$ is a Kan fibration, then there is also such a functorial long exact sequence of pointed sets and group homomorphisms, but with the strict fiber $X_s$ in place of $F$.*

*Proof.* For each $n \geq 1$, the homotopy fiber of

$$\Omega^{n-1}(f) : \Omega^{n-1}(X) \to \Omega^{n-1}(S)$$

is naturally isomorphic to $\Omega^{n-1}(F)$, as is easily seen by uncurrying both constructions. Applying Theorem 7.1.3, we therefore obtain an exact sequence

$$\pi_n(X, x) \to \pi_n(S, s) \xrightarrow{\delta_n} \pi_{n-1}(F, x) \to \pi_{n-1}(X, x) \to \pi_{n-1}(S, s).$$

Concatenating these sequences, we get our desired long exact sequence. To conclude, we just check that $\delta_n$ is a group homomorphism when $n \geq 2$.

For this, we access $\delta_n$ a bit more concretely.

$$i : \Omega S \to F$$

the map

$$i(\ell) = (x, \ell).$$

Observe that $\pi_0(i) = \delta_1$ is the boundary map $\pi_1(S, s) \to \pi_0(F)$, immediately from the construction in Ideation 7.1.1.

Unfortunately, it is not quite true that $\delta_n$ and $\pi_{n-1}(i)$ coincide for $n > 1$. However, it is the case that

$$\delta_n = \pi_{n-1}(i) \circ \tau_n,$$

where

$$\tau_n : \pi_n(S, s) \to \pi_n(S, s)$$

is the map obtained by cyclically permuting the interval coordinates of a representative $\Delta^n_1 \to S$. By Eckmann–Hilton, $\tau_n$ is a group homomorphism, and hence $\delta_n$ is as well.

When $f$ is a Kan fibration, $X_s \hookrightarrow F$ is a homotopy equivalence, and so we conclude in that case as well. □

# Chapter 8

# Weak equivalences

## 8.1 Introduction

Let $i : A \to B$ be a strongly anodyne map. By Theorem 4.8.9, for any anima $X$, every $A$-shaped diagram

$$a : A \to X$$

extends to a diagram

$$b : B \to X,$$

and this extension is unique *up to homotopy*.

This property was incredibly convenient, and we used it many times in the previous chapters. The goal of this chapter is to study a mild weakening of this property; that is, we seek to understand morphisms $i : A \to B$ such that, for any anima $X$, diagrams $A \to X$ always extend to $B$ *up to homotopy*, with the extension being unique up to homotopy. Formally, we have the following.

**Definition 8.1.1.** A *weak equivalence* of simplicial sets is a morphism $f : A \to B$ of simplicial sets so that, for any anima $X$, the map

$$\pi_0[B, X] \to \pi_0[A, X]$$

is a bijection (recall Theorem 4.7.1 implies $[A, X]$ and $[B, X]$ are both Kan complexes, and so these $\pi_0$'s can be computed naively).

**Remark 8.1.2.** The name *weak equivalence* is inspired by Theorem 4.10.1, which says that a weak equivalence of Kan complexes is just a homotopy equivalence.

**Remark 8.1.3.** The map

$$\pi_0[B, X] \to \pi_0[A, X]$$

is surjective if and only if, for any $a : A \to X$, there is some $b : B \to X$ such that $b \circ f : A \to X$ is homotopic to $a$.

Similarly, it is injective if and only if, for any two maps $b, b' : B \to X$ such that $b \circ f \simeq b' \circ f$, we also have $b \simeq b'$.

## 8.2 Basic properties of weak equivalences

Weak equivalences actually automatically obey a stronger property.

**Proposition 8.2.1.** *Let $f : A \to B$ be a weak equivalence of simplicial sets. Then, for any Kan complex $X$, the map*

$$[B, X] \to [A, X]$$

*is a homotopy equivalence of Kan complexes.*

*Proof.* By Theorem 4.10.1, it suffices to prove that, for every simplicial set $S$, the map

$$\pi_0[S, [B, X]] \to \pi_0[S, [A, X]]$$

is a bijection. But

$$[S, [B, X]] = [S \times B, X] = [B, [S, X]]$$

and

$$[S, [A, X]] = [S \times A, X] = [A, [S, X]],$$

so this bijectivity actually follows from applying the weak equivalence property of $f$ to the Kan complex $[S, X]$. □

**Theorem 8.2.2** (2-out-of-3 rule)**.** *Let $f : A \to B, g : B \to C$ be maps of simplicial sets. If any two of $f, g$, and $gf$ are weak equivalences, then so is the third.*

*Proof.* This immediately follows from the corresponding 2-out-of-3 property for bijections of sets, if you ponder the diagram

$$\pi_0[C, X] \to \pi_0[B, X] \to \pi_0[A, X].$$

□

**Proposition 8.2.3.** *Retracts of weak equivalences are weak equivalences.*

*Proof.* Take a retract diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & A' & \xrightarrow{r} & A \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w} \\ B & \xrightarrow{j} & B' & \xrightarrow{s} & B, \end{array}$$

so $ri = \mathrm{id}_A, sj = \mathrm{id}_B$ . Assume $w'$ is a weak equivalence. Applying $\pi_0([-, X])$, we get a retract diagram in Set; but retracts of bijections are bijections, and so we conclude. □

As weak equivalences are defined by a 'mapping out' property, one might suspect they play well with colimits. They do, although one must be a little careful dealing with the 'up to homotopy' nature of the mapping property. As an example of how to handle this, we prove the following result.

**Theorem 8.2.4.** *Let*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow{i'} & Y \end{array}$$

*be a pushout diagram of simplicial sets. Assume that $i$ is a monomorphism, and $f$ is a weak equivalence. Then $f'$ is a weak equivalence.*

**Warning 8.2.5.** If we do not assume $i$ is a monomorphism, then the pushout of a weak equivalence need not be a weak equivalence.

*Proof.* Let $T$ be a Kan complex. We wish to prove that

$$\pi_0[Y, T] \to \pi_0[B, T]$$

is a bijection.

For surjectivity, take $\varphi : B \to T$ some morphism. As $f$ is a weak equivalence, we can find some map $\psi : X \to T$ such that $\psi|_A \simeq \varphi|_A$.

As $i$ is an inclusion, we can lift the homotopy $\psi|_A \simeq \varphi|_A$ to some homotopy $\psi' \simeq \varphi$, where $\psi' : B \to T$ is some extension of $\psi|_A$. As our diagram is a pushout, the pair $(\psi, \psi')$ defines a map $Y \to T$ whose restriction to $B$ is homotopic to $\varphi$, as desired.

For injectivity, assume that $\varphi_0, \varphi_1 : Y \to T$ are two maps whose restrictions to $B$ are homotopic, say by a homotopy $H : B \times \Delta_1 \to T$.

To prove that $\varphi_0, \varphi_1$ are themselves homotopic, observe that $H_A := H|_{A \times \Delta_1}$ is a homotopy from $(\varphi_0)|_A$ to $(\varphi_1)|_A$. As $f$ is a weak equivalence, we can therefore find a homotopy from $(\varphi_0)|_X$ to $(\varphi_1)|_X$.

In fact, even more strongly, we know

$$f^* : [X, T] \to [A, T]$$

is a homotopy equivalence of Kan complexes; therefore it induces an equivalence on fundamental groupoids. In particular, there is some path $\tilde{H}$ from $(\varphi_0)|_X$ to $(\varphi_1)|_X$ so that $\tilde{H}|_{A \times \Delta_1}$ is homotopic rel endpoints to $H_A$.

As $A \hookrightarrow B$ is an inclusion, the map $[B, T] \to [A, T]$ is a Kan fibration. Set $S \subseteq \Delta_1 \times \Delta_1$ the simplicial subset spanned by all four vertices and the three edges $(0,0) \to (1,0), (0,0) \to (0,1)$, and $(1,0) \to (1,1)$. This inclusion $S \hookrightarrow \Delta_1^2$ is anodyne, because $S$ is actually

$$(\Delta_1 \times \{0\}) \coprod_{\{0,1\} \times \{0\}} (\{0,1\} \times \Delta_1).$$

Therefore we may solve the lifting problem

$$\begin{array}{ccc} S & \longrightarrow & [B, T] \\ \downarrow & & \downarrow \\ \Delta_1^2 & \longrightarrow & [A, T], \end{array}$$

where the bottom map is a homotopy rel endpoints from $H_A$ to $\tilde{H}_A$, and the top map $S \to [B, T]$ is from the diagram

$$\begin{array}{ccc} (\varphi_0)|_B & & (\varphi_1)|_B \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow \\ (\varphi_0)|_B & \xrightarrow{H} & (\varphi_1)|_B. \end{array}$$

The $(0,1) \to (1,1)$ edge of the resulting diagram is some homotopy $\tilde{H}_B$ from $(\varphi_0)|_B$ to $(\varphi_1)|_B$ which strictly lifts $\tilde{H}_A$. Thus we may glue it to the homotopy $\tilde{H}$ on $X$ to get a homotopy from $\varphi_0$ to $\varphi_1$, proving the desired injectivity. □

## 8.3 Some examples of weak equivalences

**Proposition 8.3.1.** *Every strongly anodyne map is a weak equivalence.*

*Proof.* Let $i : A \to B$ be strongly anodyne. Then, for every Kan complex $X$, the map

$$\theta : [B, X] \to [A, X]$$

is a trivial Kan fibration, and hence a homotopy equivalence; thus it induces a bijection on $\pi_0$, so we conclude. □

**Theorem 8.3.2.** *The map $\Delta_n \to \mathrm{pt}$ is a homotopy equivalence; in particular, for any Kan complex $X$, every map $\sigma : \Delta_n \to X$ is nullhomotopic.*

*Proof.* It suffices to create a homotopy

$$H : \Delta_1 \times \Delta_n \to \Delta_n$$

from the identity map to the constant map with value $n$. For this, we just define $H$ the monotone map

$$H(i,j) = \begin{cases} j & i = 0, \\ n & i = 1. \end{cases}$$

□

We can get a bit more leverage out of Theorem 8.3.2 with the following result.

**Proposition 8.3.3.** *If $f : X \to Y$ and $f' : X' \to Y'$ are weak equivalences, then so is*

$$f \times f' : X \times X' \to Y \times Y'.$$

*Proof.* Let $T$ be a Kan complex. The map

$$[Y \times Y', T] \to [X \times X', T]$$

is isomorphic to the composite map

$$[Y, [Y', T]] \to [Y, [X', T]] \to [X, [X', T]].$$

As $[X', T]$ is a Kan complex, by the 2-out-of-3 rule, the composite induces a bijection on $\pi_0$ if and only if $[Y, [Y', T]] \to [Y, [X', T]]$ does.

We know that $[Y', T] \to [X', T]$ is a homotopy equivalence of Kan complexes, though, so Theorem 4.10.1 applies and we conclude. □

To end, we give a large family of weak equivalences which is often useful in practice.

**Proposition 8.3.4.** *Let $A$ be a non-empty finite 1-dimensional simplicial set, and suppose $A$ is connected and cycle free (when viewed as an undirected graph – so we even forbid cycles which don't agree with edge orientations). Then any inclusion* $\mathrm{pt} \hookrightarrow A$ *is a weak equivalence.*

*Proof.* By the 2-out-of-3 rule, it is equivalent to prove $A \to \mathrm{pt}$ is a weak equivalence.

We prove this by induction on the number of non-degenerate edges of $A$. In the case of 0 edges, $A$ is a point and the claim is obvious; now the inductive step.

This $A$ is a *tree* in the language of graph theory. We claim first that $A$ has a vertex $a$ incident to exactly one edge; such a vertex $a$ is called a *leaf* in the language of graph theory.

To see why, just choose a path in the graph (so, ignoring orientations of edges) of maximal length; its endpoint must be a leaf. Such a path exists because this is a cycle-free graph, and so each path can have length at most the number of vertices in $A$ (which we assumed was finite).

Let $e$ be the edge incident to $a$, and let $b$ be the other endpoint of $e$; then we may write

$$A = A' \coprod_{\{b\}} e$$

for $A'$ the simplicial subset of $A$ obtained by deleting $a$ and $e$. As the inclusion of a vertex into $\Delta_1$ is strongly anodyne, Theorem 8.2.4 proves that $A' \hookrightarrow A$ is a weak equivalence; by the 2-out-of-3 rule and the induction hypothesis, we conclude that $A \to \mathrm{pt}$ is a weak equivalence. □

## 8.4 Weak equivalences and inclusions

If a map $i : A \to B$ is a weak equivalence, then any map $A \to X$ (for $X$ a Kan complex) extends, up to homotopy, to a map $B \to X$. And if $i : A \to B$ is an inclusion, then any map $A \to X$ which extends up to homotopy extends strictly, thanks to Corollary 5.4.3 applied to the Kan fibration $[B, X] \to [A, X]$. Thus, for $i : A \hookrightarrow B$ a weak equivalence and an inclusion, any map $A \to X$ extends *strictly* to a map $B \to X$. Generalizing this to Kan fibrations, we get the following.

**Proposition 8.4.1.** *Let $i : A \hookrightarrow B$ be an inclusion of simplicial sets, such that $i$ is a weak equivalence. Then, for any Kan fibration $f : X \to S$ where $S$ (and hence $X$) is a* Kan complex*, all lifting problems of the form*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & S \end{array}$$

*can be solved.*

*Proof.* Let $f : X \to S$ be a Kan fibration. Consider the Kan fibration

$$\theta : [B, X] \to [B, S] \times_{[A,S]} [A, X].$$

It suffices to prove $\theta$ is a trivial Kan fibration, because our lifting problem can be rephrased as lifting $\emptyset \hookrightarrow \mathrm{pt}$ against $\theta$.

Observe that the fiber product is the base change of the Kan fibration

$$\theta_S : [B, S] \to [A, S]$$

along $[A, X] \to [A, S]$. As $i$ is a weak equivalence, the Kan fibration $\theta_S$ is a homotopy equivalence of Kan complexes by Proposition 8.2.1. Hence it is a trivial Kan fibration by Corollary 7.4.2.

As trivial Kan fibrations are defined by a mapping property, they are stable under base change, and so

$$p : [B, S] \times_{[A,S]} [A, X] \to [A, X]$$

is *also* a homotopy equivalence of Kan complexes. By the 2-out-of-3 rule, as $p\theta : [B, X] \to [A, X]$ is a homotopy equivalence, we deduce $\theta$ is a trivial Kan fibration, and hence we conclude. □

## 8.5 Fibrant replacement, redux

At this point, we have proven many very useful results about anima – for example, our results about homotopy fibers. Unfortunately, none of those results apply to arbitrary simplicial sets. To port those results over, we prove that every simplicial set is weakly equivalent to a Kan complex. In fact, we can even do this in families, giving a simplicial set version of our earlier fibrant replacement result.

**Theorem 8.5.1** (Fibrant replacement)**.** *Let $f : X \to Y$ be a morphism of simplicial sets. Then, there exists*

1. *a simplicial set $Q(f)$,*
2. *a strongly anodyne map $i : X \hookrightarrow Q(f)$,*
3. *and a Kan fibration $f' : Q(f) \to Y$,*

*such that $f' \circ i = f$.*

*In fact, we can even do this functorially. That is, there is a functor*

$$Q : \mathrm{Fun}([1], \mathrm{sSet}) \to \mathrm{Fun}([2], \mathrm{sSet})$$

*sending $f : X \to Y$ to $X \xrightarrow{i} Q(f) \xrightarrow{f'} Y$ with $i, f'$ obeying the properties as above. This functor $Q$ can be chosen such that it commutes with filtered colimits.*

**Remark 8.5.2.** When $X, Y$ are both Kan complexes, Theorem 5.5.1 produced a very similar factorization. Theorem 5.5.1 has the advantage of being very explicit, while the construction we give here is very hard to actually compute – but Theorem 5.5.1 only applied to Kan complexes, whereas this theorem applies to arbitrary simplicial sets.

*Proof.* We want $Q(f) \to Y$ to be a Kan fibration. We will do this by attaching enough new simplices to $X$ that we can solve every horn filling problem.

To that end, we construct a functor

$$W : \mathrm{Fun}([1], \mathrm{sSet}) \to \mathrm{Fun}([2], \mathrm{sSet})$$

as follows. For a morphism $f : X \to Y$, let $S$ denote the set of all commutative diagrams of the form

$$\begin{array}{ccc} \Lambda^i_n & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta_n & \longrightarrow & Y, \end{array}$$

with $i, n$ varying. Note that $S$ is actually a set, because there are only set-many maps $\Delta_n \to Y$ for any $n$, and only set-many maps $\Lambda^i_n \to X$ for any $(i, n)$. For every $s \in S$, let $H_s$ denote the corresponding horn $\Lambda^i_n$, and $D_s$ the corresponding simplex $\Delta_n$.

Define $W(f)$ to be the simplicial set obtained from the pushout diagram

$$\begin{array}{ccc} \coprod_{s\in S} H_s & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{s\in S} D_s & \longrightarrow & W(f). \end{array}$$

Observe $X \to W(f)$ is strongly anodyne, as the pushout of a strongly anodyne map is always strongly anodyne.

Each $s \in S$ gives us the data of a commutative diagram

$$\begin{array}{ccc} H_s & \longrightarrow & X \\ \downarrow & & \downarrow f \\ D_s & \longrightarrow & Y. \end{array}$$

The maps $D_s \to Y$ thus combine with $f$ to induce a map

$$f' : W(f) \to Y$$

obeying $f'|_X = f$.

Hence we get our functor $W$, sending $f$ to the factorization $X \hookrightarrow W(f) \to Y$ where $X \hookrightarrow W(f)$ is strongly anodyne, and $W(f) \to Y$ is closer to being a Kan fibration than $X \to Y$ was.

Moreover, $f \mapsto W(f)$ is compatible with *filtered* colimits. This is because whenever $f$ arises as a filtered colimit of maps $f_\alpha : X_\alpha \to Y_\alpha$, then in any commutative diagram

$$\begin{array}{ccc} \Lambda^i_n & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta_n & \longrightarrow & Y, \end{array}$$

we can find some index $\alpha$ so that the diagram comes from a commutative diagram

$$\begin{array}{ccc} \Lambda^i_n & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow f_\alpha \\ \Delta_n & \longrightarrow & Y_\alpha. \end{array}$$

Indeed, any $n$-simplex of $Y$ comes from some stage $Y_{\alpha'}$, and as our colimit is *filtered*, any map $\Lambda^i_n \to X$ factors through some $X_{\alpha''}$, by taking the $(n-1)$-simplices making up $\Lambda^i_n$, choosing some $X_\beta$ they factor through, and then dominating all the $\beta$. Choose some $\alpha$ dominating $\alpha', \alpha''$, and large enough to observe the commutativity, and we conclude; as $\Lambda^i_n$ has only finitely many non-degenerate simplices, knowing our colimit is filtered allows us to check commutativity.

This functor $W$ is *almost* the functor $Q$ desired by our theorem. Unfortunately, $f' : W(f) \to Y$ *is not* always a Kan fibration: we are only able to solve horn filling problems

$$\begin{array}{ccc} \Lambda^i_n & \longrightarrow & W(f) \\ \downarrow & & \downarrow{\scriptstyle f'} \\ \Delta_n & \longrightarrow & Y \end{array}$$

where $\Lambda^i_n \to W(f)$ factors through $X$.

However, we can *iterate* $W$ to fix this. Set $W^0 = X$, $W^1 = W(f)$, and $W^n := W(W^{n-1} \to Y)$. Then we have a sequence of inclusions

$$X = W^0 \hookrightarrow W^1 \hookrightarrow W^2 \hookrightarrow \cdots .$$

Set $Q(f) := \operatorname{colim}_n W^n$.

We get a factorization

$$X \hookrightarrow Q(f) \to Y,$$

functorial in $f$.

The inclusion $X \hookrightarrow Q(f)$ is strongly anodyne, as strongly anodyne maps are stable under $\omega$-compositions.

Moreover, the map $Q(f) \to Y$ is a Kan fibration: any horn $\Lambda^i_n \to Q(f)$ factors through some $W^m$, as the horn only has finitely many non-degenerate simplices and $Q(f)$ is a filtered colimit. Thus we can solve the resulting horn lifting problem upon passage to $W^{m+1}$.

Lastly, we check that our functor $Q$ commutes with filtered colimits. This is just because we created it as a filtered colimit of repeated applications of $W$, and $W$ commutes with filtered colimits. □

**Remark 8.5.3.** This argument is often referred to as Quillen's *small object argument*.

When dealing with anima, fibrant replacement allowed us to reduce problems about functors to problems about Kan fibrations. For arbitrary simplicial sets, fibrant replacement allows us to reduce problems about simplicial sets to problems about Kan complexes. As an example, we prove the following stability property of weak equivalences.

**Theorem 8.5.4.** *Let $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ be a filtered system of weak equivalences.*

*Then the resulting map $f : X := \operatorname{colim}_\alpha X_\alpha \to \operatorname{colim}_\alpha Y_\alpha =: Y$ is a weak equivalence as well.*

*Proof.* We first assume that every $Y_\alpha$ is a Kan complex. By the functoriality of fibrant replacement, we can define a directed system of simplicial sets $\{X'_\alpha\}_\alpha$ and maps

$$X_\alpha \to X'_\alpha \to Y_\alpha$$

of directed systems such that

1. $X_\alpha \to X'_\alpha$ is anodyne,
2. $X'_\alpha \to Y_\alpha$ is a Kan fibration.

In particular, $X'_\alpha$ is a Kan complex (as $Y_\alpha$ is).

The maps $X_\alpha \to X'_\alpha$ are weak equivalences, and the composite $X_\alpha \to Y_\alpha$ is a weak equivalence, so by the 2-out-of-3 rule, each map $f'_\alpha : X'_\alpha \to Y_\alpha$ is a weak equivalence. It is also a Kan fibration of Kan complexes, and hence a trivial Kan fibration; so we can solve any lifting problem of an inclusion against $f'_\alpha$.

It follows that, on $f' : X' \to Y$, we can solve all lifting problems against inclusions of finite simplicial sets (as maps out of finite simplicial sets will factor through some particular stage of the filtered colimit). Using the skeletal filtration to build every simplicial set out of boundary inclusions $\partial\Delta_n \hookrightarrow \Delta_n$, it follows that $f'$ is a trivial Kan fibration and therefore a homotopy equivalence.

By the 2-out-of-3 rule, to prove that $f$ is a weak equivalence, it suffices to show that $X \to X'$ is; but $X \to X'$ is strongly anodyne, and hence a weak equivalence, as fibrant replacement commutes with filtered colimits, and so $X \to X' \to Y$ is actually the fibrant replacement of $f$.

Thus we've proven the theorem when each $Y_\alpha$ is a Kan complex. We now explain how to reduce to that case. By fibrant replacement, we may find for each $\alpha$ an anodyne inclusion $i : Y_\alpha \hookrightarrow Y'_\alpha$ with $Y'_\alpha$ a Kan complex.

Note that $Y' := \mathrm{colim}_\alpha Y'_\alpha$ is therefore a Kan complex (any horn inclusion factors through some finite stage, so fill it there). The map $f : X \to Y$ is a weak equivalence if and only if the composition $X \to Y \to Y'$ is, by the 2-out-of-3 rule, since $Y \to Y'$ is a weak equivalence (as it is a filtered colimit of weak equivalences all of whose targets are Kan complexes!). So, it suffices to show that $X \to Y'$ is a weak equivalence, but now $Y'$ is a colimit of Kan complexes, so we conclude. □

A second wonderful corollary of fibrant replacement is that every anodyne morphism is actually strongly anodyne.

**Proposition 8.5.5.** *An anodyne morphism $i : A \to B$ of simplicial sets is always strongly anodyne.*

*Proof.* By fibrant replacement, we can write $i$ as a composition of maps

$$j : A \hookrightarrow A'$$

and

$$f : A' \to B,$$

where $f$ is a Kan fibration, and $j$ is strongly anodyne.

As $i$ is anodyne, we can solve the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{j} & A' \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

to find a map $s : B \to A'$ such that

$$f \circ s = \mathrm{id}_B,$$

$$s \circ i = j.$$

The commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{s} & A' & \xrightarrow{f} & B \end{array}$$

witnesses $i$ as a retract of $j$; as strongly anodyne maps are stable under retracts, we conclude. □

## 8.6 Geometric realization of weak equivalences

The geometric realization of any weak equivalence is a homotopy equivalence.

**Proposition 8.6.1.** *Let $f : X \to Y$ be a weak equivalence of simplicial sets. Then $|f|$ is a homotopy equivalence of point-set topological spaces.*

*Proof.* The simplicial set $\mathrm{Sing}\,|X|$ is a Kan complex; thus the map

$$[Y, \mathrm{Sing}\,|X|] \to [X, \mathrm{Sing}\,|X|]$$

is a homotopy equivalence. In particular, there is some simplicial set map $g : Y \to \mathrm{Sing}\,|X|$ so that $gf$ is homotopic to the unit map $\eta_X : X \to \mathrm{Sing}\,|X|$.

Let $\bar{g} : |Y| \to |X|$ be the adjunct of $g$, so that $\bar{g} = \epsilon_{|X|}|g|$.

By Corollary 4.5.20, it follows that $\bar{g} \circ |f|$ is homotopic to $\mathrm{id}_{|X|}$.

Next, we must check that $|f| \circ \bar{g} : |Y| \to |Y|$ is homotopic to $\mathrm{id}_{|Y|}$. This map $|f| \circ \bar{g}$ is adjunct to the composite map

$$\alpha : Y \xrightarrow{\eta_Y} \mathrm{Sing}\,|Y| \xrightarrow{\mathrm{Sing}(|f| \circ \bar{g})} \mathrm{Sing}(|Y|).$$

Again by Corollary 4.5.20, it suffices to check that $\alpha$ is homotopic to the unit map $\eta_Y : Y \to \mathrm{Sing}\,|Y|$. Observe that

$$\alpha \circ f = \mathrm{Sing}(|f| \circ \bar{g}) \circ \eta_Y \circ f = \mathrm{Sing}(|f| \circ \bar{g}) \circ \mathrm{Sing}(|f|) \circ \eta_X,$$

where we use that the unit of adjunction is a natural transformation to rewrite $\eta_Y \circ f$ as $\mathrm{Sing}(|f|) \circ \eta_X$.

Write

$$\mathrm{Sing}(|f| \circ \bar{g}) \circ \mathrm{Sing}(|f|) = \mathrm{Sing}(|f|) \circ \mathrm{Sing}(\bar{g} \circ |f|).$$

We know that $\bar{g} \circ |f|$ is homotopic to $\mathrm{id}_{|X|}$ already; take some homotopy

$$H : |X| \times [0, 1] \to |X|$$

from $\bar{g}|f|$ to $\mathrm{id}_{|X|}$. Then define

$$H' : \mathrm{Sing}(|X|) \times \Delta_1 \to \mathrm{Sing}(|X|)$$

as the composite map

$$\mathrm{Sing}(|X|) \times \Delta_1 \to \mathrm{Sing}(|X|) \times \mathrm{Sing}(|\Delta_1|) = \mathrm{Sing}(|X| \times [0, 1]) \xrightarrow{\mathrm{Sing}(H)} \mathrm{Sing}(|X|),$$

and this $H'$ will be a homotopy from $\mathrm{Sing}(\bar{g} \circ |f|)$ to $\mathrm{id}$. Thus

$$\mathrm{Sing}(|f| \circ \bar{g}) \circ \mathrm{Sing}(|f|) \simeq \mathrm{Sing}(|f|),$$

so that

$$\alpha \circ f \simeq \mathrm{Sing}(|f|) \circ \eta_X = \eta_Y \circ f.$$

As $f$ is a weak equivalence and $\mathrm{Sing}\,|Y|$ is a Kan complex, it follows that $\alpha \simeq \eta_Y$, as desired. □

# Chapter 9

# Whitehead's theorem

## 9.1 Introduction

It is often important in mathematics to check that a function of sets is a bijection; similarly, it is often of importance in groupoid-based mathematics to check that a functor between anima is an equivalence.

Unfortunately for groupoid-based mathematics, to check something is an equivalence requires, at the moment, one to construct a homotopy inverse. In general there are many homotopy inverses (the inverse is only unique up to homotopy), which means constructing a specific homotopy inverse requires making some arbitrary choice – in mathematics it is often the case that it is harder to construct flexible objects than it is rigid ones, and so constructing homotopy inverses is sometimes quite tricky.

In set theory, though, there is a way to prove a function of sets is a bijection without constructing an inverse: first prove it is injective, and then prove it is surjective. In other words, you show all the fibers of your function are points.

In $\infty$-groupoid theory, the analogous statement would be that one can check a functor is an equivalence by showing each of its *homotopy fibers* is *homotopy equivalent* to the point.

**Definition 9.1.1.** An $\infty$-groupoid $X$ is said to be *contractible* if $X \to \mathrm{pt}$ is a homotopy equivalence.

In this chapter, we prove *Whitehead's theorem*, telling us that indeed we can check whether a map of anima is an equivalence just by checking if its homotopy fibers are contractible.

Whitehead's theorem has two parts: first showing that the homotopy fibers of a homotopy equivalence are contractible, and then showing that the converse holds. We start by showing homotopy fibers are contractible.

**Remark 9.1.2.** By fibrant replacement, it is enough to prove Whitehead's theorem for Kan fibrations; so, for now we will focus on the case of Kan fibrations, and at the end we give the general argument which uses fibrant replacement to reduce to this Kan fibration case.

**Ideation 9.1.3.** How might we show the fibers of a trivial Kan fibration are contractible? By the long exact sequence of a fibration, all the higher homotopy groups of a fiber of a trivial Kan fibration vanish. So, to begin, we study $\infty$-groupoids all of whose higher homotopy groups vanish.

## 9.2 Higher connectivity

**Ideation 9.2.1.** Let $(X, x)$ be a pointed Kan complex which is connected, and all of whose homotopy groups vanish. What can we deduce about $X$?

The first observation is that $\pi_{\leq 1}(X)$ is equivalent to the trivial groupoid. This implies two things:

1. any two maps $\Delta_1 \to X$ with the same endpoints are homotopic rel their endpoints,
2. any map $\partial\Delta_2 \to X$ extends to a map $\Delta_2 \to X$.

Thus, when $\pi_{\leq 1}(X)$ is trivial, we have two sorts of properties: a *homotopy uniqueness* property for extensions of maps $\partial\Delta_1 \to X$ to $\Delta_1$, and an *existence* property for extending maps $\partial\Delta_2 \to X$ to $\Delta_2$.

At first glance, this claim might not seem so interesting; but because simplicial sets are built out of simplices by making boundary attachments, this claim is actually incredibly powerful: if $A \hookrightarrow B$ is *any* inclusion of simplicial sets where $\dim(B) \leq 2$, then any map $A \to X$ extends to a map $B \to X$, by an easy Hahn-Banach style argument. Incorporating the higher homotopy groups, we find the following.

**Definition 9.2.2.** A pointed Kan complex $(X, x)$ is said to be *$n$-connective* if

$$\pi_i(X, x) = \mathrm{pt}$$

for all $0 \leq i \leq n$.

**Example 9.2.3.** In particular, $(X, x)$ is 0-connective if and only if it is connected, and 1-connective if and only if $\pi_{\leq 1}(X)$ is equivalent to the trivial groupoid.

**Theorem 9.2.4.** *Let $(X, x)$ be a pointed Kan complex, and $n \geq 0$ a natural number. Then the following are equivalent:*

1. *$X$ is $n$-connective,*
2. *for every $i \leq n+1$, every map $\partial\Delta_i \to X$ extends to $\Delta_i$,*
3. *for every $i \leq n$, every map $\partial\Delta_i \to X$ extends to $\Delta_i$, and any two such extensions are homotopic rel $\partial\Delta_i$,*
4. *for any simplicial set $B$ with $\dim(B) \leq n+1$, and $A \hookrightarrow B$ any simplicial subset, every map $A \to X$ extends to a map $B \to X$.*

**Remark 9.2.5.** This result fits into a common pattern: a *uniqueness* result in one dimension is equivalent to an *existence* result in the next dimension up.

*Proof.* Assume first that the third condition holds. To begin, we prove a slightly stronger version of the third condition, and then use that to deduce the second condition. We wish to prove the second condition holds; so, take

$$a : \partial\Delta_{n+1} \to X$$

a map we wish to extend to $\Delta_{n+1}$. As

$$\partial\Delta_{n+1} = \Lambda^0_{n+1} \coprod_{\partial\Delta_n} \Delta_n,$$

we can view $a$ as a pair of maps

$$a_0 : \Lambda^0_{n+1} \to X,$$
$$a_1 : \Delta_n \to X$$

which agree on $\partial\Delta_n$.

As $\Lambda^0_{n+1} \hookrightarrow \Delta_{n+1}$ and $\mathrm{pt} \hookrightarrow \Delta_{n+1}$ are both weak equivalences, the 2-out-of-3 rule implies $\mathrm{pt} \hookrightarrow \Lambda^0_{n+1}$ is a weak equivalence as well, and so $a_0$ is nullhomotopic. Choose

$$H_0 : \Lambda^0_{n+1} \times \Delta_1 \to X$$

a homotopy from $a_0$ to the constant map with value $x$. The restriction of $H_0$ to $\partial\Delta_n$ gives a homotopy

$$\tilde{H} : \partial\Delta_n \times \Delta_1 \to X$$

from $(a_1)|_{\partial\Delta_n}$ to the constant map of value $x$.

Let

$$\theta_n : [\Delta_n, X] \to [\partial\Delta_n, X]$$

be the restriction map. This $\theta_n$ is a Kan fibration whose fibers are connected; by the long exact sequence of a fibration, the hypotheses of Theorem 7.4.1 apply, and so we can lift $\tilde{H}$ to a homotopy

$$H_1 : \Delta_n \times \Delta_1 \to X$$

from $a_1$ to the constant map with value $x$.

These homotopies $H_0, H_1$ then glue to a homotopy from $a$ to the constant map with value $x$. In particular, $a$ is nullhomotopic. Thus $[\partial\Delta_{n+1}, X]$ is connected. As the Kan fibration

$$\theta_{n+1} : [\Delta_{n+1}, X] \to [\partial\Delta_{n+1}, X]$$

obviously has at least one non-empty fiber (for example, the fiber above any $n+1$ simplex of $X$), we deduce that all of its fibers are non-empty.

Next, assume condition 2 holds; we will use it to prove condition 4. We do this by an easy Hahn–Banach style argument. Given an inclusion $A \hookrightarrow B$ with $\dim(B) \leq n+1$, and a map $a : A \to X$, let $\mathcal{L}$ denote the poset of pairs $(a', A')$ where $A \subseteq A' \subseteq B$ and $a'$ extends $a$, with $\mathcal{L}$ ordered by extension. By Zorn's lemma, this poset admits a maximal element $(a', A')$. If $A' \neq B$, then let $\sigma \in B_i$ be some non-degenerate simplex of $B$ not contained in $A'$, but of minimal dimension (so all of its boundary faces lie in $A'$). We can easily use condition 2 to extend $a'$ to $A' \coprod_{\partial\Delta_i} \Delta_i$, contradicting the maximality of $(a', A')$. Thus condition 4 holds.

Assume condition 4 holds; we now prove condition 3 holds. Given two maps

$$b_0, b_1 : \Delta_i \to X$$

with $i \leq n$, whose restrictions

$$a_0, a_1 : \partial\Delta_i \to X$$

are *equal*, we wish to prove that the maps $b_0, b_1$ are homotopic rel $\partial\Delta_i$.

This is equivalent to solving a lifting problem against

$$(\partial\Delta_1 \times \Delta_i) \coprod_{\partial\Delta_1 \times \partial\Delta_i} (\Delta_1 \times \partial\Delta_i) \hookrightarrow \Delta_1 \times \Delta_i,$$

which of course condition 4 says we may do.

Thus two implies four implies three implies two, and so the last three conditions are equivalent.

To show condition 1 implies condition 2, we induct on $n$. When $n = 0$, the equivalence between conditions 1 and 2 is clear.

For the inductive step, assume that $X$ is $n$-connective. To prove condition 3, it suffices to show that if $b, b' : \Delta_n \to X$ are two maps which agree on $\partial\Delta_n$, then they are homotopic rel $\partial\Delta_n$. This is equivalent to a path lifting problem in

$$\theta : [\Delta_n, X] \to [\partial\Delta_n, X].$$

As $\dim(\partial\Delta_n) = n - 1$, we find $\dim \Delta_1 \times \partial\Delta_n = n$. So, by condition 4 for $n - 1$, we deduce that $[\partial\Delta_n, X]$ is connected.

The fibers of $\theta$ are thus all homotopy equivalent; but

$$\pi_n(X, x) \simeq \pi_0[\Delta_n/\partial\Delta_n, (X, x)]_* \tag{9.2.6}$$

is the set of connected components of one fiber of $\theta$, and so all fibers of $\theta$ are connected. We conclude.

(9.2.6) shows that condition 3 implies condition 1, and so we conclude that all four properties are equivalent. □

**Corollary 9.2.7** (Whitehead's theorem, absolute case)**.** *Let $(X, x)$ be a pointed Kan complex. Suppose that $\pi_n(X, x) = \mathrm{pt}$ for all $n \geq 0$. Then the map*

$$(X, x) \to \mathrm{pt}$$

*is a trivial Kan fibration.*

*Proof.* By Theorem 9.2.4, we can extend any map $\partial\Delta_i \to X$ to a map $\Delta_i \to X$. By an easy Hahn-Banach style argument à la the proof of Theorem 9.2.4, we conclude. □

## 9.3 Relative $n$-connectivity

We can give a relative variant of the notion of $n$-connectivity.

**Definition 9.3.1.** A morphism $f : X \to S$ of simplicial sets is said to be *$n$-connective*, for $n \geq 0$, if every lifting problem

$$\begin{array}{ccc} \partial\Delta_i & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta_i & \longrightarrow & S \end{array}$$

with $i \leq n + 1$ has a solution.

**Theorem 9.3.2.** *Let $f : X \to S$ be a Kan fibration of anima, and $n \geq 0$ a natural number. Then the following are equivalent:*

1. *$f$ is $n$-connective,*
2. *for any simplicial set $B$ with $\dim(B) \leq n+1$, and $A \hookrightarrow B$ any simplicial subset, every lifting problem*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow f \\ B & \longrightarrow & S \end{array}$$

   *admits a solution,*
3. *every fiber $X_s$ of $f$ is nonempty and $n$-connective.*

*Proof.* Condition 1 implies condition 2 by an easy Hahn–Banach argument as in the proof of Theorem 9.2.4. Condition 2 obviously implies condition 1, and so the first two conditions are equivalent.

Condition 1 implies condition 3 by Theorem 9.2.4, as this lifting property is stable under base change.

Conversely, assume condition 3 holds. To prove condition 1, it is enough to show that, for every $i \leq n+1$, all fibers of the Kan fibration

$$\theta_i : [\Delta_i, X] \to [\partial\Delta_i, X] \times_{[\partial\Delta_i, S]} [\Delta_i, S]$$

are nonempty.

Let $(a : \partial\Delta_i \to X, \bar{b} : \Delta_i \to S)$ be a point of the target of $\theta_i$.

As $S$ is an anima and $\Delta_i$ is weakly equivalent to the point, any map $\bar{b}$ is nullhomotopic. Choose $\bar{H}$ a homotopy from $\bar{b}$ to a constant map; using that $[\partial\Delta_i, X] \to [\partial\Delta_i, S]$ is a Kan fibration, we can lift $\bar{H}$ to a homotopy $H$ whose left endpoint is $a$.

As the fiber above a point of a Kan fibration depends (up to homotopy) only on the connected components of the base the point is in, it suffices to prove that all fibers of $\theta_i$ above points $(a, \bar{b})$ with $\bar{b}$ constant are nonempty; but then we use Theorem 9.2.4 to conclude, as now our lifting problem becomes an extension problem in $X_s$. □

## 9.4 Whitehead's theorem for functors

We can finally fulfill the promise of the introduction.

**Theorem 9.4.1** (Whitehead's theorem for anima)**.** *Let $f : X \to S$ be a functor of $\infty$-groupoids. Then, the following are equivalent:*

1. *$f$ is a homotopy equivalence,*
2. *every homotopy fiber of $f$ is a contractible Kan complex,*
3. *the map $\pi_0(X) \to \pi_0(S)$ is a bijection, and for every $n \geq 1$ and basepoint $x \in X_0$, the map $\pi_n(X, x) \to \pi_n(S, f(x))$ is a group isomorphism.*

*Proof.* Assume first that $f$ is a Kan fibration.

In this case, conditions 2 and 3 are obviously equivalent by the long exact sequence of a fibration and Whitehead's theorem in the absolute case (Corollary 9.2.7). Condition 1 also implies condition 3, by the homotopy invariance of homotopy groups.

Assume now that condition 2 holds. By Theorem 9.3.2, $f$ is $n$-connective for all $n$, and so by a Hahn–Banach style argument it is easy to see that $f$ is a trivial Kan fibration, and hence condition 1 holds.

Conversely, condition 1 implies $f$ is a trivial Kan fibration, and so all of its base changes are; thus condition 2 holds.

For the general case, we use fibrant replacement; we can then factor $f$ as

$$X \xrightarrow{i} X' \xrightarrow{f'} S,$$

where $X'$ is the path space, $i$ is a homotopy equivalence, and $f'$ is a Kan fibration. The strict fibers of $f'$ are the homotopy fibers of $f$.

By the 2-out-of-3 rule, $f$ is a homotopy equivalence if and only if $f'$ is. Thus condition 1 for $f$ and $f'$ are equivalent.

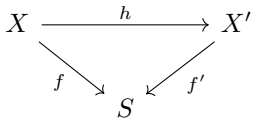
Condition 2 for $f$ is equivalent to condition 2 for $f'$, as the homotopy fibers of $f$ are the strict fibers of $f'$, which are homotopic to the homotopy fibers of $f'$.

As $i$ is a homotopy equivalence, condition 3 for $f$ is equivalent to condition 3 for $f'$ by homotopy invariance of higher homotopy groups. Thus the Kan fibration case implies the general case. □

## 9.5 Families of homotopy equivalences

As an application of Whitehead's theorem, in this section we study families of homotopy equivalences.

**Ideation 9.5.1.** Consider a commutative diagram




where $f, f'$ are both Kan fibrations (so that $X, X'$ are "relative to $S$" $\infty$-groupoids).

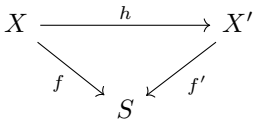
We imagine this as some family of functors $h_s : X_s \to X'_s$ over $S$.

Imagine $S$ is an $\infty$-groupoid, to reduce technicalities. Suppose $h$ is a homotopy equivalence. Is each $h_s$ a homotopy equivalence? Conversely, if each $h_s$ is a homotopy equivalence, is $h$?

These questions both have positive answers when, instead of equivalences of anima, one works with bijections of sets; and in fact, it is often useful to check that a function $h$ is a bijection by looking at slices $h_s$, since each slice $h_s$ might be individually easier to study.

The goal of this section is to use Whitehead's theorem to give the $\infty$-groupoid versions of these results.

**Proposition 9.5.2.** *Let*



*be a commutative diagram where $f, f'$ are both Kan fibrations and $S$ is a Kan complex.*

*Then $h$ is a homotopy equivalence if and only if, for each $s \in S_0$, the map $h_s : X_s \to X'_s$ is a homotopy equivalence.*

*Proof.* Assume first that $h$ is a homotopy equivalence; the trouble is that the homotopy inverse of $h$ has no reason to be compatible with the maps to $S$, and so it is not immediately clear how to study $h_s$.

However, on $\pi_0$, everything is OK; that is, we claim $\pi_0(h_s)$ is a bijection for each $s$. To see this, take $x' \in X'_s$ some vertex. Choose $h' : X' \to X$ a homotopy inverse of $h$.

Then $x := h'(x')$ is such that $h(x)$ lies in the same path component of $X'$ as $x'$. Let $\tilde{p} : h(x) \to x'$ be a path between them.

Set $p = f'\tilde{p} : f(x) = f'h(x) \to f'(x') = s$ be the image of $\tilde{p}$ in $S$. As $f$ is a Kan fibration, we can find some path $q : x \to \tilde{x}$ such that $f(q) = p$. In particular, $\tilde{x}$ lies in $X_s$.

We claim that $h(\tilde{x})$ lies in the same path component of $X'_s$ as $x'$. For this, observe that $\tilde{p}$ and $hq$ are both lifts of $p$ to $X'$ with left endpoint $h(x)$. By homotopy uniqueness of solutions to lifting problems, we can then find a homotopy rel left endpoints from $\tilde{p}$ to $hq$; this homotopy then gives us a path in $X'_s$ from $x'$ to $h(\tilde{x})$, proving our desired surjectivity.

For injectivity, it suffices to assume $X_s \neq \emptyset$; choose $x \in X_s$ a basepoint, and put $x' = h(x)$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\pi_1(X,x) & \longrightarrow & \pi_1(S,s) & \longrightarrow & \pi_0(X_s,x) & \longrightarrow & \pi_0(X,x) & \longrightarrow & \pi_0(S,s) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_1(X',x') & \longrightarrow & \pi_1(S,s) & \longrightarrow & \pi_0(X'_s,x') & \longrightarrow & \pi_0(X',x') & \longrightarrow & \pi_0(S,s)
\end{array}$$

with exact rows; the vertical map $\pi_0(X_s, x) \to \pi_0(X'_s, x')$ is surjective, and all other maps are bijections.

To deduce injectivity of $\pi_0(h_s)$, assume $\pi_0(h_s)([x_1]_{X_s}) = \pi_0(h_s)([x]_{X_s})$ (it suffices to check injectivity above $x$ by varying our original choice of basepoint). Then $h[x_1]_X = h[x]_X$, so (as $\pi_0(h)$ is a bijection) we deduce that $[x_1]_{X_s}$ lies in the image of $\pi_1(S,s) \to \pi_0(X_s, x)$. Say $[\ell] \in \pi_1(S,s)$ has $\delta([\ell]) = [x_1]_{X_s}$.

Then, in the bottom exact sequence, $\delta'([\ell]) = [x']_{X'_s}$, implying $[\ell]$ lies in the image of $\pi_1(X,x) \to \pi_1(S,s)$. But then exactness of the top row gives us that $\delta([\ell]) = [x]_{X_s}$, and so we conclude injectivity.

From here, a similar five lemma argument proves $\pi_n(h_s)$ is a bijection for all $n \geq 1$, and so we conclude that if $h$ is a homotopy equivalence, then each $h_s$ is.

Conversely, assume each $h_s$ is a homotopy equivalence. Again the five lemma proves $\pi_n(h)$ is a bijection for all $n \geq 1$. By Whitehead's theorem, all that remains is to prove $\pi_0(h)$ is a bijection.

Surjectivity is immediate (if $x' \in X'$, use surjectivity of $\pi_0(h_{f'(x')})$). For injectivity, assume that $x_0, x_1 \in X$ are such that $[hx_0]_{X'} = [hx_1]_{X'}$. Choose a path $p : hx_0 \to hx_1$ between them.

Lift $f'p$ to get $q : x_0 \to \tilde{x_1}$ in $X_1$ with $f(\tilde{x}_1) = f(x_1)$. Then $hq$ and $p$ are two different lifts of $f'p$ with the same left endpoint, so they are homotopic rel their left endpoint; this homotopy gives us a path from $h\tilde{x}_1$ to $hx_1$ in $X'_{f(x_1)}$. As $\pi_0(h_{f(x_1)})$ is a bijection, it follows that $\tilde{x}_1, x_1$ lie in the same path component of $X$, and so by transitivity we deduce that $x_0, x_1$ lie in the same path component of $X$. □

When dealing with $\infty$-groupoids, homotopy equivalences ought to be just as good as equalities. In that vein, it is natural to ask for a variant of Proposition 9.5.2 where, instead of having two Kan fibrations over a common base, we have the following slightly more general setup.

**Theorem 9.5.3.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ S & \xrightarrow{h} & S' \end{array}$$

*be a commutative diagram of Kan complexes, in which $f, f'$ are Kan fibrations and $h$ is a homotopy equivalence.*

*Then $g$ is a homotopy equivalence if and only if, for every $s \in S_0$, the map $X_s \to X'_{h(s)}$ induced by $g$ is a homotopy equivalence.*

*Proof.* By fibrant replacement, we may factor $h$ as

$$S \xrightarrow{i} \tilde{S} \xrightarrow{\tilde{h}} S'$$

where $i : S \to \tilde{S}$ is strongly anodyne and $\tilde{h}$ is a Kan fibration. By the 2-out-of-3 rule, as $h$ is a homotopy equivalence we find that $\tilde{h}$ is in fact a trivial Kan fibration. As trivial Kan fibrations are stable under base change, $X' \times_{S'} \tilde{S} \to X'$ is also a trivial Kan fibration. Set $\tilde{X} = X' \times_{S'} \tilde{S}$.

The map $X \to X'$ factors through $\tilde{X}$, and so by the 2-out-of-3 rule, it is equivalent to work with $X \to \tilde{X}$ instead of $X \to X'$. This construction also preserves the fibers, as $\tilde{X}_{i(s)} = X'_{h(s)}$. Thus it suffices to prove this theorem in the case where the homotopy equivalence $h$ is also strongly anodyne.

In this case, set $Y = X' \times_{S'} S$, and let $q : Y \to X'$ be the projection.

To start, we prove the map $q$ is a weak equivalence; this is in fact a very special case of the main theorem. In this case, we can give a very explicit argument. As $h$ is strongly anodyne and $S$ is Kan, we can find a retraction $r : S' \to S$ with $rh = \mathrm{id}_S$ by solving a lifting problem. The maps $\mathrm{id}_{S'}$ and $hr$ agree on $S$, so by Theorem 4.8.9 they are homotopic; in fact, we can even find a homotopy

$$H : S' \times \Delta_1 \to S'$$

from $\mathrm{id}_{S'}$ to $hr$ which is constant on $S$. Thus $r$ is a homotopy inverse to $h$.

To conclude that $q$ is a homotopy equivalence, we need only find a homotopy

$$\tilde{H} : X' \times \Delta_1 \to X'$$

from $\mathrm{id}_{X'}$ to $qR$. As $h$ is an inclusion, so is $q$; thus

$$(Y \times \Delta_1) \coprod_{Y \times \{0\}} (X' \times \{0\}) \hookrightarrow X' \times \Delta_1$$

is anodyne. Solving a lifting problem against the Kan fibration $f'$, we can find a homotopy

$$K : X' \times \Delta_1 \to X'$$

with left endpoint $\mathrm{id}_{X'}$, whose restriction to $Y \times \Delta_1$ is the constant homotopy from $\mathrm{id}_Y$ to itself, and whose projection to $S'$ is $H$.

In particular,

$$f' \circ K_1 = hr \circ f'.$$

Thus we may define $R : X' \to Y$ by

$$R(x') = (K_1(x'), r(f'(x'))),$$

and $Rq = \mathrm{id}_Y$ . The map $K$ is a homotopy from $\mathrm{id}_{X'}$ to $qR$, implying that $q, R$ are homotopy inverse; thus $q$ is a homotopy equivalence, as desired.

By the 2-out-of-3 rule, it follows that $g$ is a homotopy equivalence if and only if the natural map $X \to Y$ is. Applying Proposition 9.5.2, we conclude. □

# Chapter 10

# Homotopy pullbacks and pushouts

## 10.1 Introduction

In Chapter 5, we discussed the theory of homotopy fibers in quite a bit of detail. While toward the end we focused more on the topological intuition, our original motivation was from the theory of higher groupoids: we wanted a variant of the usual fiber which was better behaved homotopically.

In category theory, the ordinary fiber is a special case of the more general fiber product. Thus it is natural to ask if there is a notion of *homotopy fiber products* in general. Inspired by our definition of the homotopy fiber, we make the following definition.

**Definition 10.1.1** (Homotopy fiber product)**.** Let $f : X \to S, g : Y \to S$ be two morphisms of anima. The *homotopy fiber product* of $f$ and $g$ is the simplicial set

$$X \times^{\mathrm{h}}_S Y := X \times_{[\{0\},S]} [\Delta_1, S] \times_{[\{1\},S]} Y.$$

**Remark 10.1.2.** That is, while a point of $X \times_S Y$ is a point of $X$ and a point of $Y$ who have the same image in $S$, a point of $X \times^{\mathrm{h}}_S Y$ is a point $x \in X$, a point $y \in Y$, and a *reason* that $f(x) \simeq g(y)$.

**Warning 10.1.3.** While this definition makes sense for arbitrary simplicial sets, it turns out to only be well-behaved when $S$ is a Kan complex.

## 10.2 Basic properties of the homotopy fiber product

To begin, we check that the homotopy fiber product of anima is still an anima.

**Proposition 10.2.1.** *Let $f : X \to S, g : Y \to S$ be two morphisms of anima. Then the simplicial set $X \times^{\mathrm{h}}_S Y$ is an anima.*

*In fact, something stronger is true: the map*

$$X \times^{\mathrm{h}}_S Y \to X \times Y$$

*is a Kan fibration.*

*Proof.* Take a map $\varphi : \Lambda^i_n \to X \times^{\mathrm{h}}_S Y$. This is the data of maps

$$\varphi_X : \Lambda^i_n \to X,$$

$$\varphi_Y : \Lambda^i_n \to Y,$$

and a homotopy

$$H : \Lambda^i_n \times \Delta_1 \to S$$

from $f \circ \varphi_X$ to $g \circ \varphi_Y$.

Suppose we are given maps

$$\psi_X : \Delta_n \to X,$$

$$\psi_Y : \Delta_n \to Y$$

which extend $\varphi_X, \varphi_Y$.

By Theorem 4.8.8, we may extend $H$ to a homotopy from $f\psi_X$ to $g\psi_Y$. Thus we get a map

$$\psi : \Delta_n \to X \times^{\mathrm{h}}_S Y$$

extending $\varphi$ and compatible with $\psi_X, \psi_Y$. Thus $X \times^{\mathrm{h}}_S Y \to X \times Y$ is a Kan fibration, as desired. □

The Kan fibration of Proposition 10.2.1 allows us to understand the homotopy groups of $X \times^{\mathrm{h}}_S Y$.

**Proposition 10.2.2.** *Let $f : X \to S, g : Y \to S$ be morphisms of anima. Then the fiber of*

$$p : X \times^{\mathrm{h}}_S Y \to X \times Y$$

*above $(x, y)$ is isomorphic to the path space $P_{f(x),g(y)}$ (of $S$).*

**Remark 10.2.3.** This makes intuitive sense: a point of $X \times^{\mathrm{h}}_S Y$ is a point of $X$, a point of $Y$, and a reason that those points have equivalent images in $S$; this path space is that torsor of reasons.

*Proof.* A map $T \to p^{-1}(x, y)$ is, by definition, the data of a homotopy

$$H : T \times \Delta_1 \to S$$

from the constant map $T \to S$ with value $f(x)$ to the constant map with value $g(y)$. This is exactly the universal property of the path space, and so we conclude. □

**Corollary 10.2.4.** *Let $f : X \to S, g : Y \to S$ be morphisms of anima. Suppose $x \in X_0, y \in Y_0$ are two points with $f(x) = g(y) = s$. Then there is a long exact sequence of pointed sets*

$$\cdots \to \pi_1(X \times Y, (x, y)) \to \pi_1(S, s) \to \pi_0(X \times^{\mathrm{h}}_S Y, (x, c_s, y)) \to \pi_0(X \times Y, (x, y)).$$

For Kan fibrations, homotopy fibers and strict fibers coincide (up to homotopy). It is then tempting to ask if the same holds with general fiber products; and fortunately it does, for basically the same reason.

**Proposition 10.2.5.** *Let $f : X \to S$ be a Kan fibration of anima, and let $g : Y \to S$ be an arbitrary morphism of anima. Then the natural map*

$$p : X \times_S Y \to X \times^{\mathrm{h}}_S Y$$

*is a homotopy equivalence.*

*Proof.* As $f$ is a Kan fibration, by Theorem 5.5.1 the map

$$f' : P \to S, P = X \times_{[0,S]} [\Delta_1, S]$$

is also a Kan fibration, and $i : X \hookrightarrow P$ is a homotopy equivalence.

The relevance of this replacement is that $X \times^{\mathrm{h}}_S Y = P \times_S Y$, so we can rewrite $p$ as a map

$$p : X \times_S Y \hookrightarrow P \times_S Y.$$

We will write down an explicit homotopy inverse to $p$. Intuitively, we want this inverse

$$r : P \times_S Y \to X \times_S Y$$

to send a triple $(x, p, y)$ to the pair $(\tilde{p}(1), y)$, where $\tilde{p}$ is some lift of $p$.

As $[\Delta_1, X] \to P$ is a trivial Kan fibration by Theorem 4.8.13, we can solve a lifting problem against the inclusion $i : X \hookrightarrow P$ to find a map

$$\ell : P \to [\Delta_1, X]$$

sending $(x, p)$ to a lift of $p : \Delta_1 \to S$ to a map $\tilde{p} : \Delta_1 \to X$, with $\tilde{p}(0) = x$, and so that $\ell i(x) = c_x$ is the constant path.

With $\ell$ in hand, we define

$$r : P \times_S Y \to X \times_S Y$$

as the map whose projection onto the $Y$ factor is the obvious projection, and whose projection onto the $X$ factor is the composition

$$P \times_S Y \to P \xrightarrow{\ell} [\Delta_1, X] \xrightarrow{\mathrm{eval}_1} X.$$

These maps obviously agree over $S$, and hence give us our $r$.

Note that $r \circ p = \mathrm{id}$, because $\ell i(x) = c_x$, and so $r$ is a left inverse of $p$. We can define a homotopy

$$H : (P \times_S Y) \times \Delta_1 \to P \times_S Y$$

from id to $pr$, by the formula

$$H((x, \gamma, y), t) = (\tilde{\gamma}(t), s \mapsto \gamma(\max(t, s)), y).$$

□

Finally, we check that homotopy fiber products are symmetric in the two arguments.

**Proposition 10.2.6.** *Let $f : X \to S, g : Y \to S$ be two morphisms of anima. Then there is a homotopy equivalence*

$$X \times^{\mathrm{h}}_S Y \simeq Y \times^{\mathrm{h}}_S X.$$

*Proof.* We can use a zigzag. Let $Pf, Pg$ be the resulting path spaces; then

$$X \times^{\mathrm{h}}_S Y = Pf \times_S Y,$$

$$Y \times^{\mathrm{h}}_S X = Pg \times_S X,$$

and both spaces have natural maps to $Pf \times_S Pg$. We show that, in both cases, the natural maps to $Pf \times_S Pg$ are homotopy equivalences.

For $Pf \times_S Y$, observe that there is a Cartesian diagram

$$\begin{array}{ccc} Pf \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle i_Y} \\ Pf \times_S Pg & \longrightarrow & Pg. \end{array}$$

Unfortunately, even though $i_Y$ is a homotopy equivalence, it is not a Kan fibration, and hence its pullbacks are not always homotopy equivalences; we still have something to check. However, Theorem 9.5.3 can be applied, as both horizontal maps are Kan fibrations; thus to prove that our map is a homotopy equivalence, it suffices to check that its fibers are homotopy equivalences. But in fact the fibers are isomorphisms and so we conclude.

Similarly, $X \times_S Pg \to Pf \times_S Pg$ is a homotopy equivalence. Observe that $Pf \times_S Pg, Pf \times_S Y$, and $X \times_S Pg$ are all anima; for example observe that $Pg \to S$ is a Kan fibration and so $X \times_S Pg \to X$ is Kan as well, implying (as $X$ is an anima) that $X \times_S Pg$ is an anima. Thus the homotopy equivalences we've produced imply the existence of the desired homotopy equivalence, and so we conclude. □

## 10.3 Homotopy pushouts

In usual category theory, the dual concept to fiber product is the pushout. Is there a homotopy pushout?

In Set, to form a pushout $X \coprod_A Y$, one forms the quotient $(X \coprod Y)/\sim$, where we quotiented by a certain equivalence relation. But the *whole point* of anima was that *quotients by equivalence relations are lossy*. Really, we should remember that each element $a \in A$ gives a 'reason' to identify some point of $X$ with a point of $Y$.

In our Kan complex model of anima, 'reasons' are paths. This suggests the following notion of a *homotopy pushout*.

**Ideation 10.3.1.** Take $f : A \to X, g : A \to Y$ two morphisms of simplicial sets. The homotopy pushout should be some simplicial set which formalizes the idea of "take a disjoint union of $X$ and $Y$, but for each $a \in A$, attach a path from $f(a)$ to $g(a)$."

A path ought to mean $\Delta_1$. The space of 'one path for every point of $A$' is just the product $A \times \Delta_1$ (often called the *cylinder* on $A$).

**Definition 10.3.2** (Direct homotopy pushouts)**.** Let $f : A \to X, g : A \to Y$ be two morphisms of simplicial sets. The *homotopy pushout* of $f, g$ is the simplicial set $X \coprod^{\mathrm{h}}_A Y$ given as the colimit of

$$\begin{array}{ccccc} A \times \{0\} & \longrightarrow & A \times \Delta_1 & \longleftarrow & A \times \{1\} \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle g} \\ X & & & & Y. \end{array}$$

**Remark 10.3.3.** One should visualize this as follows: take a cylinder with base $A$, and then attach $X, Y$ to the opposite ends.

**Warning 10.3.4.** Even when $A, X, Y$ are all anima, their direct homotopy pushout has no reason to be an anima! For this reason, we have to be quite careful when dealing with homotopy pushouts in the work below; our notes on $\infty$-*categories* will give us a better way to handle homotopy pushouts.

The homotopy pushout can be described quite concretely.

**Proposition 10.3.5.** *Let $f : A \to X, g : A \to Y$ be two morphisms of simplicial sets. Then*

$$(X \overset{\mathrm{h}}{\coprod_A} Y)_n = X_n \coprod Y_n \coprod (A_n \times \{\alpha : [n] \to [1] \mid \alpha \text{ nonconstant}\}).$$

*Proof.* By definition, an $n$-simplex of $X \coprod_A^{\mathrm{h}} Y$ is obtained as

$$(X_n \coprod (A_n \times (\Delta_1)_n) \coprod Y_n)/\sim,$$

where we use the equivalence relation generated by

$$f(a) \sim (a, c_0), g(a) \sim (a, c_1),$$

where $c_0, c_1$ are the constant maps.

It is easy to see that the resulting equivalence relation just has classes

$$\{x\} \cup \{(a, c_0) \mid f(a) = x\},$$

$$\{y\} \cup \{(a, c_1) \mid g(a) = y\},$$

$$\{(a, p)\},$$

where in the last class $p \notin \{c_0, c_1\}$. Our claimed equality follows immediately. □

Homotopy fiber products along Kan fibrations are easy; similarly, homotopy pushouts along inclusions are easy to compute.

**Construction 10.3.6** (Cylinder collapse map)**.** Let $f : A \to X, g : A \to Y$ be morphisms of simplicial sets. We construct a map

$$c : X \overset{\mathrm{h}}{\coprod_A} Y \to X \coprod_A Y,$$

called the *cylinder collapse map.*

We map out of the colimit diagram used to define the homotopy pushout. On $X, Y$, let $c$ be the natural maps into the pushout. On $A \times \Delta_1$, let it be the map

$$A \times \Delta_1 \to A \xrightarrow{f} X \to X \coprod_A Y,$$

where the compatibility with $A \times \{1\} \to Y$ is because $i_X f = i_Y g$.

**Proposition 10.3.7.** *Let $i : A \hookrightarrow B$ be an inclusion of simplicial sets and $f : A \to X$ a morphism. Then the cylinder collapse map*

$$c : B \overset{\mathrm{h}}{\coprod_A} X \to B \coprod_A X$$

*is a weak equivalence of simplicial sets.*

*Proof.* To begin, we build the homotopy pushout in two stages. To start, let's ignore $X$ entirely, and just attach $B$ to the bottom face of a cylinder on $A$. The resulting simplicial set is

$$C := (A \times \Delta_1) \coprod_{A \times \{0\}} (B \times \{0\}).$$

As $A \hookrightarrow B$ is an inclusion and $\{0\} \hookrightarrow \Delta_1$ is anodyne, the map

$$j : C \hookrightarrow B \times \Delta_1$$

is anodyne, and therefore a weak equivalence. This is a good start! To get the homotopy pushout from $C$, we next attach $X$ along $A \times \{1\}$, resulting in the pushout diagram

$$\begin{array}{ccc} A \times \{1\} & \longrightarrow & C \\ \downarrow{\scriptstyle f} & & \downarrow \\ X & \longrightarrow & B \coprod_A^{\mathrm{h}} X. \end{array}$$

However, there's a certain trick we can do here: let's enlarge the homotopy pushout by replacing the cylinder on $A$ with one along all of $B$. In other words, we form the pushout

$$C' := (B \times \Delta_1) \coprod_{A \times \{1\}} X.$$

The inclusion $j$ gives us a map $j' : B \coprod_A^{\mathrm{h}} X \to C'$. In fact, this map $j'$ is therefore obtained from the pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & B \coprod_A^{\mathrm{h}} X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ B \times \Delta_1 & \longrightarrow & C', \end{array}$$

and so $j'$ is anodyne (as it is the pushout of an anodyne morphism). In particular, $j'$ is a weak equivalence.

But $c$ factors through $C'$, via the map

$$c' : C' \to B \coprod_A X$$

which on $B \times \Delta_1$ is just projection onto $B$ followed by the canonical map $B \to B \coprod_A X$, and which on $X$ is the obvious map.

By the 2-out-of-3 rule, it follows that $c$ is a weak equivalence if and only if $c'$ is. Thus we need to check that the obvious map

$$c' : (B \times \Delta_1) \coprod_{A \times \{1\}} X \to B \coprod_A X$$

is a weak equivalence. In fact, it is a homotopy equivalence; define

$$s : B \coprod_A X \to (B \times \Delta_1) \coprod_{A \times \{1\}} X,$$

$$s(b) = (b, 1); s(x) = x.$$

Then $c' \circ s = \mathrm{id}$. In the other direction, $s \circ c' \simeq \mathrm{id}$, by the homotopy

$$H : C' \times \Delta_1 \to C',$$

$$H(b, t, s) = (b, \max(t, s)); H(x, s) = x.$$

We conclude. □

**Ideation 10.3.8.** How did we compute homotopy fibers? Usually the trick was to take a fibrant replacement to write our map as a Kan fibration (at least up to homotopy equivalence), and then use that homotopy fibers of Kan fibrations could be computed as strict fibers.

Using Proposition 10.3.7, we can hope to compute (at least up to weak equivalence) all homotopy pushouts as strict pushouts. However, we need a slight modification of fibrant replacement: we need to write all maps as an inclusion composed with a homotopy equivalence. Fortunately, it's actually not so hard to do this.

**Construction 10.3.9** (Mapping cylinders)**.** Let $f : X \to Y$ be a morphism of simplicial sets. The *mapping cylinder* of $f$ is the simplicial set

$$Mf := (X \times \Delta_1) \coprod_{X \times \{1\}} Y.$$

We can factor $f$ as the inclusion

$$i : X \hookrightarrow Mf$$

(the inclusion is onto $X \times \{0\}$) followed by the homotopy equivalence

$$f' : Mf \to Y,$$

which on $X \times \Delta_1$ is given by projecting to $X$ and then doing $f$, and which on $Y$ is identity. To see $f'$ is a homotopy equivalence, observe that it admits a section

$$s : Y \to Mf, s(y) = y,$$

and in fact there's a canonical homotopy

$$H : Mf \times \Delta_1 \to Mf$$

from $\mathrm{id}_{Mf}$ to $sf'$, given by

$$H(x, t, s) = (x, \max(t, s)), H(y, s) = y.$$

**Ideation 10.3.10.** Even with mapping cylinders, we are not yet done with the idea proposed in Ideation 10.3.8. If $f : A \to X, g : A \to Y$ are two maps we wish to compute the homotopy pushout of, then factor $f$ as

$$A \hookrightarrow Mf \xrightarrow{f'} X.$$

This leads to a commutative diagram

$$\begin{array}{ccccc} Mf & \xleftarrow{i} & A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xleftarrow{f} & A & \xrightarrow{g} & Y, \end{array}$$

giving us a map of homotopy pushouts

$$Mf \overset{\mathrm{h}}{\coprod_A} Y \to X \overset{\mathrm{h}}{\coprod_A} Y.$$

We must prove that this map is a weak equivalence.

**Lemma 10.3.11.** *Suppose*

$$\begin{array}{ccccc} X & \xleftarrow{f} & A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle u_X} & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X' & \xleftarrow{f'} & A & \xrightarrow{g} & Y \end{array}$$

*is a commutative diagram in which $u_X$ is a weak equivalence. Then the induced map*

$$\alpha : X \overset{\mathrm{h}}{\coprod_A} Y \to X' \overset{\mathrm{h}}{\coprod_A} Y$$

*is a weak equivalence as well.*

*Proof.* Observe that $\alpha$ fits into a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X \coprod_A^{\mathrm{h}} Y \\ \downarrow{\scriptstyle u_X} & & \downarrow{\scriptstyle \alpha} \\ X' & \longrightarrow & X' \coprod_A^{\mathrm{h}} Y. \end{array}$$

By Theorem 8.2.4, to prove $\alpha$ is a weak equivalence it suffices to prove that $X \to X \coprod_A^{\mathrm{h}} Y$ is an inclusion. Fortunately, this is always true for homotopy pushouts (as follows from, say, Proposition 10.3.5). We conclude. □

Lemma 10.3.11 is technically enough for Ideation 10.3.10 (and hence for the easy computation of homotopy pushouts up to weak equivalence), but of course it is a special case of a more general homotopy invariance statement. This more general statement can be proved with a quick devissage, so we give it.

**Proposition 10.3.12.** *Suppose*

$$\begin{array}{ccccc} X & \xleftarrow{f} & A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle u_X} & & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle u_Y} \\ X' & \xleftarrow{f'} & A' & \xrightarrow{g'} & Y' \end{array}$$

*is a commutative diagram in which $u_X, u_A, u_Y$ are all weak equivalences. Then the induced map*

$$X \overset{\mathrm{h}}{\coprod_A} Y \to X' \overset{\mathrm{h}}{\coprod_{A'}} Y'$$

*is a weak equivalence as well.*

*Proof.* We can treat this 'one change at a time,' by factoring our diagram as

$$\begin{array}{ccccc}
X & \xleftarrow{f} & A & \xrightarrow{g} & Y \\
\downarrow{\scriptstyle u_X} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
X' & \xleftarrow{u_X f} & A & \xrightarrow{g} & Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle u_Y} \\
X' & \xleftarrow{u_X f} & A & \xrightarrow{u_Y g} & Y' \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle \mathrm{id}} \\
X' & \xleftarrow{f'} & A' & \xrightarrow{g'} & Y'.
\end{array}$$

Using Lemma 10.3.11 on $u_X, u_Y$, we can therefore assume that $u_X = u_Y = \mathrm{id}$ in our argument, so that we're only changing $A$. Thus our diagram takes the form

$$\begin{array}{ccccc}
X & \xleftarrow{f} & A & \xrightarrow{g} & Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xleftarrow{f'} & A' & \xrightarrow{g'} & Y,
\end{array}$$

implying $f = f'u_A, g = g'u_A$.

To conclude, fix a test Kan complex $T$. The mapping space $[X \coprod^{\mathrm{h}}_A Y, T]$ is easy to describe: it is just

$$E := [X \overset{\mathrm{h}}{\coprod_A} Y, T] = B \times_{C \times C} [\Delta_1, C],$$

where $B = [X, T] \times [Y, T]$ and $C = [A, T]$. That is, a map out of the homotopy pushout is just the data of two maps $X \to T, Y \to T$ together with a homotopy of their restrictions to $A$. Setting $C' = [A', T]$, we similarly find

$$E' := [X \overset{\mathrm{h}}{\coprod_{A'}} Y, T] = B \times_{C' \times C'} [\Delta_1, C'].$$

As $u_A$ is a weak equivalence, $u_A^* : C' \to C$ is a homotopy equivalence. Then $E, E'$ are both Kan fibrations over $B$, whose fibers above $(\alpha, \beta)$ are the path spaces $\mathrm{Mor}_C(\alpha f, \beta g)$ and $\mathrm{Mor}_{C'}(\alpha f', \beta g')$. As $C, C'$ are homotopy equivalent, those mapping spaces are homotopy equivalent; it follows from Proposition 9.5.2 that $E' \to E$ is therefore a homotopy equivalence. As this holds for every Kan complex $T$, we conclude. □

**Corollary 10.3.13.** *Let $f : A \to X, g : A \to Y$ be two simplicial set maps, and let $Mf$ be the mapping cylinder of $f$. Then the natural maps*

$$Mf \overset{\mathrm{h}}{\coprod_A} Y \to X \overset{\mathrm{h}}{\coprod_A} Y,$$

$$Mf \overset{\mathrm{h}}{\coprod_A} Y \to Mf \coprod_A Y$$

*are both weak equivalences.*

# Part III

# Advanced properties of simplicial sets

# Chapter 11

# Barycentric subdivision

## 11.1 Introduction

When trying to use simplicial sets to model point-set topological spaces, we frequently encounter the following situation.

We have two simplicial sets $S$ and $T$, with geometric realizations $X = |S|$ and $Y = |T|$. Then we write down a continuous function $f : X \to Y$, which we'd like to study using simplicial sets. However, this $f$ often does not arise as the geometric realization of any map $S \to T$, which is incredibly inconvenient.

**Example 11.1.1.** Consider the simplicial set $S = \Delta_1/\partial\Delta_1$. Then $|S|$ is a circle. Let $f : |S| \to |S|$ denote the map $z \mapsto z^2$. This is a continuous endomorphism of the circle which is not homotopic to the identity and not homotopic to a constant map. However, the only simplicial set maps $S \to S$ are the constant map and the identity!

There are two (related) ways to solve the problem of Example 11.1.1; generally, given a continuous map $f : |S| \to |T|$ that we want to come from a geometric realization, we could either

1. add more simplices to $S$, so that each simplex of $S$ is small enough that $f$ maps it to a simplex of $T$,
2. add more singular simplices to $T$, to account for the images of simplices of $S$ under $f$.

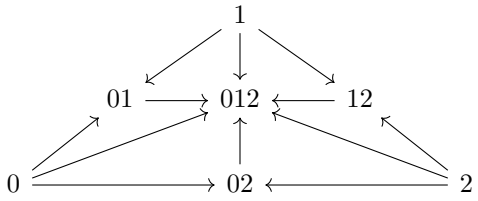
The first solution is more geometrically intuitive, and easier to understand.

**Example 11.1.2.** Continuing Example 11.1.1, let $S'$ be the simplicial set obtained by bisecting the one edge of $S$. More precisely, $S'_0 = \{a, b\}$ and $S'_1 = \{\ell_1, \ell_2, s(a), s(b)\}$, where $\ell_1$ goes from $a$ to $b$ and $\ell_2$ goes from $b$ to $a$. Then $|S'|$ is a circle, and we can easily define a simplicial set map $\varphi : S' \to S$ which has $|\varphi| \simeq f$.

Elaborating on this first solution leads to the concept of *barycentric subdivision*, a procedure for creating more simplices in a simplicial set. This chapter explores barycentric subdivision in much detail.

## 11.2 Barycentric subdivision

*Barycentric subdivision* replaces simplices of a simplicial set with finer ones, by adjoining the barycenter of every simplex as a new vertex, and then connecting them. For example, the barycentric subdivision of $\Delta_2$ is the simplicial set




Note that we labelled the barycenter of each simplex by the list of all the vertices in the simplex we took the barycenter of. This generalizes quite well, leading to the following definition.

**Definition 11.2.1** (The cosimplicial object of barycenters)**.** We define a cosimplicial object $B^\bullet$ of sSet as follows.

In degree $n$, put

$$B^n := N(C_n),$$

where $C_n$ denotes the poset of all *nonempty* subsets of $\{0, 1, ..., n\}$, ordered by inclusion.

For a morphism $f : [n] \to [m]$ of totally ordered sets, we define a corresponding map $B^n \to B^m$ as

$$N(C_n) \xrightarrow{N(C_f)} N(C_m),$$

where $C_f : C_n \to C_m$ sends a chain $S \subseteq [n]$ to $f(S) \subseteq [m]$.

This makes $B^\bullet$ into a cosimplicial object.

**Definition 11.2.2.** We define

$$\mathrm{sd} : \mathrm{sSet} \to \mathrm{sSet}$$

as $\| - \|_{B^\bullet}$, for $B^\bullet$ the cosimplicial object of Definition 11.2.1. We refer to sd as the *subdivision functor*.

It can be helpful to describe $\mathrm{sd}(X)$ more explicitly.

**Ideation 11.2.3.** Explicitly,

$$\mathrm{sd}(X) = \mathrm{colim}_{(m,\sigma\in X_m)} B^m.$$

Thus, to give an $n$-simplex of $\mathrm{sd}(X)$, one needs to give an integer $m$, a simplex $\tau \in X_m$, and a chain

$$\emptyset \neq S_0 \subseteq \cdots \subseteq S_n \subseteq [m]$$

of subsets of $[m]$.

In this colimit, we identify two triples when they belong to the same equivalence class of the smallest equivalence relation generated by declaring $(m, \tau, S_\bullet)$ and $(m', \tau', S'_\bullet)$ to be equivalent whenever we can find a monotone map $\varphi : [m] \to [m']$ such that $\tau = \varphi^*\tau'$ and $S'_i = \varphi(S_i)$. We call such identifications the *elementary identifications*.

Is there some 'simplest' triple representing a given $n$-simplex?

Given a triple $(m, \tau, S_\bullet)$, the monotone inclusion

$$\alpha : S_n \hookrightarrow [m],$$

witnesses that $(m, \tau, S_\bullet)$ and $(|S_n| - 1, \alpha^*\tau, S_\bullet)$ represent the same simplex. Thus we can always find a triple representing an $n$-simplex which has $m = |S_n| - 1$, essentially by deleting 'irrelevant' entries of $[m]$.

Take $(m, \tau, S_\bullet)$ then a triple with $S_n = [m]$. By Proposition 2.2.9, we can also write $\tau = \pi^*\tau'$ for some surjective monotone map $\pi : [m] \to [m']$, with $\tau'$ a non-degenerate simplex. Then $(m', \tau', \pi(S_\bullet))$ is (by surjectivity of $\pi$) a representative of $(m, \tau, S_\bullet)$ also having $\pi(S_n) = [m']$ but with $\tau'$ non-degenerate. Thus we can always represent an $n$-simplex of $\mathrm{sd}(X)$ by a triple $(m, \tau, S_\bullet)$ where both $\tau$ is non-degenerate and $S_n = [m]$.

This leads to the following.

**Lemma 11.2.4.** *Let $X$ be a simplicial set. Then, in the colimit*

$$\mathrm{sd}(X)_n = \mathrm{colim}_{(m,\sigma\in X_m)} B^m_n,$$

*every $n$-simplex has a* unique *representative of the form $(m, \sigma \in X_m, S_\bullet)$ where*

1. *$\sigma$ is non-degenerate,*
2. *$S_n = [m]$.*

**Notation 11.2.5.** The unique representative of an $n$-simplex having $S_n = [m]$ and $\sigma$ non-degenerate is called the *canonical representative*, and the corresponding triple $(m, \sigma, S_\bullet)$ is said to be in *canonical form*.

*Proof.* Ideation 11.2.3 proves existence; in fact, Ideation 11.2.3 describes an explicit two-step procedure to turn any triple into an equivalent one in canonical form; we call the resulting triple the *normalization* of the original simplex.

For uniqueness, it suffices to prove that two triples related by an elementary identification have the same normalization. Take $S_\bullet \subseteq [m]$ some chain of subsets, $\varphi : [m] \to [m']$ a monotone map, and $\sigma' \in X_{m'}$. Set $\sigma := \varphi^*\sigma', S'_\bullet := \varphi(S_\bullet)$. Then $(m, \sigma, S_\bullet)$ and $(m', \sigma', S'_\bullet)$ are related by an elementary identification; we prove they have the same normalization.

To obtain the normalizations, we first pullback along

$$\alpha : S_n \hookrightarrow [m]$$

and

$$\alpha' : S'_n \hookrightarrow [m'].$$

We then write

$$\alpha^*\sigma = \pi^*\tau,$$
$$(\alpha')^*\sigma' = (\pi')^*\tau',$$

where $\tau, \tau'$ are nondegenerate and $\pi, \pi'$ are monotone surjections. Observe that

$$\pi^*\tau = \alpha^*\sigma = \alpha^*\varphi^*\sigma' = (\varphi|_{S_n\to S'_n})^*(\alpha')^*\sigma' = (\varphi|_{S_n\to S'_n})^*(\pi')^*\tau'.$$

As $\varphi|_{S_n\to S'_n}$ is surjective, by the uniqueness in Proposition 2.2.9 we deduce that actually $\tau = \tau'$, and

$$\pi = \pi' \circ \varphi_{S_n\to S'_n}.$$

From here it is easy to see that the normal forms are equal, as desired. □

As an example application of Lemma 11.2.4, we prove the following.

**Lemma 11.2.6.** *If $i : X \hookrightarrow Y$ is an inclusion of simplicial sets, then* $\mathrm{sd}(i)$ *is also an inclusion.*

*Proof.* The map $\mathrm{sd}(i) : \mathrm{sd}(X) \to \mathrm{sd}(Y)$ sends the $n$-simplex $(m, \sigma, S_\bullet)$ (in canonical form) of $\mathrm{sd}(X)$ to $(m, i(\sigma), S_\bullet) \in \mathrm{sd}(Y)_n$.

Observe that $i(\sigma)$ is also non-degenerate, so $(m, i(\sigma), S_\bullet)$ is also in canonical form; indeed, if $i(\sigma)$ were degenerate, then we could write $i(\sigma) = \pi^*(\tau)$ for $\pi$ a non-identity surjection and $\tau$ some simplex of $Y$; choose $s$ a section of $\pi$ (so $\pi s = \mathrm{id}$) and observe that

$$i(s^*\sigma) = \tau,$$

so by injectivity of $i$, we find $\pi^*(s^*\sigma) = \sigma$, forcing $\sigma$ to be degenerate.

We therefore get our desired injectivity by uniqueness of canonical representatives. □

As sd is defined by means of a cosimplicial set, it also has a right adjoint.

**Definition 11.2.7.** We define $\mathrm{Ex} : \mathrm{sSet} \to \mathrm{sSet}$ the functor $\mathrm{Sing}^{B^\bullet}$.

### 11.2.1 The subdivision of a poset

The subdivision of $\Delta_n$ is very easy to describe (it is $B^n$); similarly, we can describe the subdivision of the nerve of any poset.

**Definition 11.2.8.** Let $P$ be a poset. We define a new poset $\mathrm{sd}(P)$ whose elements are nonempty finite chains in $P$, ordered by inclusion.

**Construction 11.2.9.** Let $P$ be a poset. We construct a morphism

$$\eta_P : N(\mathrm{sd}(P)) \to \mathrm{sd}(N(P)),$$

functorially in $P$, as follows.

An $n$-simplex of $N(\mathrm{sd}(P))$ is a sequence

$$C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n$$

where $C_i$ is a chain in $P$. Fix such a sequence, and write $C_n = \{x_0 < x_1 < \cdots < x_m\}$. Then we define

$$S_i = \{j \mid x_j \in C_i\}$$

and put

$$\eta_P(C_\bullet) = (C_n, S_\bullet).$$

**Proposition 11.2.10.** *For any poset $P$, the map $\eta_P$ of Construction 11.2.9 is an isomorphism of simplicial sets.*

*Proof.* By Lemma 11.2.4, an $n$-simplex of $\mathrm{sd}(N(P))$ is the data of a non-degenerate $m$-simplex $\sigma$ of $N(P)$, together with a chain of subsets

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = [m].$$

This $\sigma$ is just a chain in $P$ of length $m$, and so the pair $(\sigma, S_\bullet)$ is equivalent to giving a sequence of chains

$$C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n$$

where $|C_n| = m + 1$.

But such a chain is exactly the data of an $n$-simplex of $N(\mathrm{sd}(P))$, and so $\eta_P$ is a bijection. □

### 11.2.2 The subdivision of a braced simplicial set

For a *braced* simplicial set $X$, we can describe $\mathrm{sd}(X)$ as the nerve of a certain category.

**Remark 11.2.11.** Recall that braced simplicial sets are the most combinatorial, and simplest, of all simplicial sets. This is what allows us to describe their subdivisions so concretely.

**Construction 11.2.12.** Let $X$ be a braced simplicial set. Its *category of nondegenerate simplices* is the category $P^{\mathrm{nd}}X$ whose objects are the non-degenerate simplices of $X$ (of any dimension), and where, for $\sigma \in X_n, \tau \in X_m$ both non-degenerate,

$$\mathrm{Hom}_{P^{\mathrm{nd}}X}(\sigma, \tau) = \{\varphi : [n] \to [m] \text{ injective, monotone} \mid \varphi^*\tau = \sigma\}.$$

**Remark 11.2.13.** In other words, we add morphisms indicating which simplices are faces of others.

**Construction 11.2.14.** Let $X$ be a braced simplicial set. We construct a map

$$\psi_X : N(P^{\mathrm{nd}}X) \to \mathrm{sd}(X).$$

An $n$-simplex of $N(P^{\mathrm{nd}}X)$ is a chain

$$\sigma_\bullet := \sigma_{k_0} \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-1}} \sigma_{k_n},$$

where $\sigma_{k_i}$ is a $k_i$-dimensional non-degenerate simplex of $X$, and $\varphi_i : [k_i] \to [k_{i+1}]$ is an injective monotone map with $\varphi_i^*\sigma_{k_{i+1}} = \sigma_{k_i}$.

We define

$$\psi_X(\sigma_\bullet) = (k_n, \sigma_{k_n}, S_\bullet),$$

where $S_i := \mathrm{im}([k_i] \to [k_n])$, the map being the composite $\varphi_{n-1} \circ \cdots \circ \varphi_i$.

**Theorem 11.2.15.** *Let $X$ be a braced simplicial set. Then the map*

$$\psi_X : N(P^{\mathrm{nd}}X) \to \mathrm{sd}(X)$$

*of Construction 11.2.14 is an isomorphism.*

*Proof.* Observe that the triple $(k_n, \sigma_{k_n}, S_\bullet)$ used to define $\psi_X$ is always in canonical form. As the chain $\sigma_\bullet$ is determined by $S_\bullet$ and $\sigma_{k_n}$, our map $\psi_X$ is injective; surjectivity is easy to see, as given $(k, \sigma_k, S_\bullet)$ in canonical form, we can define $\sigma_i$ as the restriction of $\sigma_k$ to $S_i$ to get a chain $\sigma_\bullet$. □

**Corollary 11.2.16.** *The subdivision of a braced simplicial set is braced.*

*Proof.* By Theorem 11.2.15, it suffices to check that when $X$ is braced, $N(P^{\mathrm{nd}}X)$ is braced. An $n$-simplex of the nerve of this category is the data of an integer $m$, a *non-degenerate* simplex $\sigma \in X_m$, and a chain of non-empty subsets $S_0 \subseteq \cdots \subseteq S_n = [m]$. This simplex is non-degenerate exactly when each inclusion is proper; as face maps are computed just by dropping terms from this chain, clearly faces of a non-degenerate simplex are still non-degenerate. □

Nerves of posets are a little simpler than nerves of categories, so the following observation is sometimes of use.

**Corollary 11.2.17.** *Let $X$ be a braced simplicial set. Then $\mathrm{sd}^2(X)$ is isomorphic to the nerve of a poset.*

*Proof.* By two applications of Theorem 11.2.15, it suffices to prove that the category $P^{\mathrm{nd}}(N(P^{\mathrm{nd}}X))$ is a poset; an $n$-simplex of $N(P^{\mathrm{nd}}X)$ is a chain

$$\sigma_0 \to \cdots \to \sigma_n$$

of composable morphisms in $P^{\mathrm{nd}}X$, and this $n$-simplex is non-degenerate exactly when no morphism is the identity. In this case, it is clear that all faces of $\sigma_\bullet$ are distinct, and so $P^{\mathrm{nd}}(N(P^{\mathrm{nd}}X))$ is a poset, as desired. □

## 11.3 The geometric realization of a subdivision

**Ideation 11.3.1.** With barycentric subdivision introduced, our 'goal theorem' is something like the following: if $A, B$ are simplicial sets and $\varphi : |A| \to |B|$ is continuous, then there is some $n > 0$ so that $\varphi$ is homotopic to the geometric realization of some simplicial set map $f : \mathrm{sd}^n(A) \to B$.

This statement doesn't even type check at the moment, though: $\varphi$ has domain $|A|$, whereas $|f|$ has domain $|\mathrm{sd}^n(A)|$. To make sense of this statement, we need to compare $|\mathrm{sd}(A)|$ to $|A|$ (and hopefully prove they are canonically homeomorphic!). Observe that $|A|$ and $|\mathrm{sd}(A)|$ are both geometric realizations of $A$, with respect to two different cosimplicial spaces: the standard cosimplicial space $\Delta^\bullet$, and the cosimplicial space $|B^\bullet|$. We start by comparing these two cosimplicial spaces.

**Lemma 11.3.2.** *The standard co-*semi*simplicial topological space $\Delta^\bullet$ is isomorphic (as a co-semisimplicial space) to the co-*semi*simplicial space $|B^\bullet|$.*

**Warning 11.3.3.** These are only isomorphic as co-semisimplicial spaces, and not as cosimplicial spaces!

*Proof.* We start by constructing continuous functions

$$\varphi_n : |B^n| \to \Delta^n.$$

The vertices of $|B^n|$ are the nonempty subsets $S$ of $[n]$. Given a nonempty subset $S$, we define

$$\varphi_n(S) = b_S,$$

where $b_S$ is the barycenter of the set $S \subseteq \Delta^n$. The space $|B^n|$ is built by glueing together the $m$-simplices of $B^n$; for each $m$-simplex $\sigma$ of $B^n$, we let $\varphi_n|_{|\sigma|}$ be the unique *affine* map interpolating between the values we prescribed on the vertices of $\sigma$.

The intuition here is that $S$ is supposed to be a vertex we add to the point $b_S$ of $\Delta^n$.

This gives us a continuous map $\varphi_n$. As $\varphi_n$ is a continuous map between two compact Hausdorff spaces, to prove it is a homeomorphism it suffices to check it is a bijection.

A point of $|B^n|$ is the data of a triple

$$(m \in \{0, ..., n\}, S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m, (s_0, ..., s_m)),$$

where the $S_i$ are nonempty subsets of $[n]$ and $(s_0, ..., s_m) \in \Delta^m$. In fact, $|B^n|$ is obtained by taking the disjoint union of copies of $\Delta^m$ parametrized by pairs $(m, S_\bullet)$, and then glueing them in the obvious way. The image of this point under $\varphi_n$ is

$$\sum_{i=0}^{m} s_i b_{S_i}.$$

If any $s_i$ is zero, then this point actually comes from an $(m-1)$-simplex of $|B^n|$, so we can always normalize to assume each $s_i$ is strictly positive.

Take $t = (t_0, ..., t_n) \in \Delta^n$. Let $t'_0 > t'_1 > \cdots > t'_m > 0$ be the set of distinct *positive* values amongst the coordinates of $t$, and put $t'_{m+1} = 0$.

$$S_i = \{j \mid t_j \geq t'_i\} = \{j \mid t_j \in \{t'_0, ..., t'_i\}\}.$$

Then

$$S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m \subseteq [n].$$

We put $s_i = |S_i| \cdot (t'_i - t'_{i+1})$. Then $\sum_i s_i = \sum_j t_j = 1$, and so $(m, S_\bullet, s)$ gives a point of $|B^n|$.

Set $\tilde{t} = \sum_{i=0}^m s_i b_{S_i}$. Then

$$\begin{aligned}
\tilde{t}_j &= \sum_{i=0}^m s_i (b_{S_i})_j \\
&= \sum_{i=0}^m \frac{s_i}{|S_i|} \cdot \delta_{j \in S_i} \\
&= \sum_{i=0}^m (t'_i - t'_{i+1}) \cdot \delta_{j \in S_i} \\
&= t_j,
\end{aligned}$$

so that $t = \tilde{t}$, as desired. This gives us surjectivity.

For injectivity, observe that if $\varphi_n(m, S_\bullet, s) = t$, then we always have

$$t_j = \sum_{\{i \mid j \in S_i\}} \frac{s_i}{|S_i|}.$$

The right hand side takes on exactly $m+1$ distinct positive values (because the subsets $S_i$ form a chain and each $s_i$ is nonzero). Thus $m+1$ is the number of distinct positive coordinates of $t$. Ordering these values as $t'_0 > \cdots > t'_m > 0$, we recover

$$S_i = \{j \mid t_j \geq t'_i\},$$

$$s_i = |S_i|(t'_i - t'_{i+1}),$$

as before.

To upgrade these homeomorphisms to an isomorphism of co-semisimplicial spaces, we check that $\varphi_n$ is compatible with the face maps; we warn the reader that it is not, in general, compatible with degeneracies! Suppose $a : [m] \to [n]$ is injective and monotone. We wish to check that the diagram

$$\begin{array}{ccc}
|B^m| & \xrightarrow{|B(a)|} & |B^n| \\
\downarrow{\scriptstyle \varphi_m} & & \downarrow{\scriptstyle \varphi_n} \\
\Delta^m & \xrightarrow{\Delta(a)} & \Delta^n
\end{array}$$

commutes. Note that the map $|B(a)|$ is affine when restricted to any simplex (as all geometric realizations of simplicial set maps are). Therefore both composite maps $|B^m| \to \Delta^n$ are affine on each simplex, and so it suffices to check that the composites are equal on the vertices.

Let $S \subseteq [m]$ be nonempty. Then

$$\varphi_n(B(a)(S)) = \varphi_n(a(S)) = b_{a(S)},$$

and

$$\Delta(a)(\varphi_m(S)) = \Delta(a)(b_S) = b_{a(S)},$$

so we conclude. □

**Warning 11.3.4.** When $a : [m] \to [n]$ is monotone but *not* injective, it is not always the case that $\Delta(a)(b_S) = b_{a(S)}$. In general,

$$\Delta(a)(b_S) = \left( \frac{|S \cap a^{-1}(0)|}{|S|}, ..., \frac{|S \cap a^{-1}(n)|}{|S|} \right).$$

When $a$ is injective, this is exactly $b_{a(S)}$. But when $a$ is not injective, this output need not be the barycenter of any face of $\Delta_n$.

Lemma 11.3.2 has two consequences, one for Ex, and one for sd .

**Theorem 11.3.5.** *Let $Y$ be a topological space. Then there is a (functorial in $Y$) isomorphism of* semi*simplicial sets*

$$\mathrm{Sing}(Y) \simeq \mathrm{Ex}(\mathrm{Sing}(Y)).$$

**Warning 11.3.6.** In general, this is only an isomorphism of semisimplicial sets!

*Proof.* Both semisimplicial sets are obtained as the singular complex functor of a co-semisimplicial space; by Lemma 11.3.2, these co-semisimplicial spaces are isomorphic, and hence the associated functors are naturally isomorphic. □

**Theorem 11.3.7.** *Let $X$ be a* braced *simplicial set. Then there is a homeomorphism*

$$|X| \simeq |\,\mathrm{sd}(X)|. \tag{11.3.8}$$

*These homeomorphisms can be chosen so that they are functorial in maps $u : X \to Y$ of braced simplicial sets which* carry non-degenerate simplices to non-degenerate simplices*.*

**Warning 11.3.9.** The isomorphisms (11.3.8) are *not functorial* for arbitrary maps of braced simplicial sets.

*Proof.* Let $U$ be the forgetful functor from simplicial sets to semi-simplicial sets. Theorem 11.3.5 asserts a natural isomorphism $U \circ \mathrm{Sing} \simeq U \circ \mathrm{Ex} \circ \mathrm{Sing}$ . Recall that $U$ is right adjoint to the construction $S \mapsto S^+$ of Proposition 2.5.9; thus the natural isomorphism of Theorem 11.3.5 becomes a natural isomorphism

$$|(-)^+| \simeq |\,\mathrm{sd}((-)^+)|$$

by uniqueness of left adjoints. In particular, for any simplicial set $X$ of the form $S^+$ with $S$ semi-simplicial, we have $|X| \simeq |\,\mathrm{sd}(X)|$. As the essential image of $S \mapsto S^+$ is precisely the class of braced simplicial sets, we conclude.

For the functoriality, note that if $u$ carries non-degenerate simplices to non-degenerate simplices, then it comes from a map of semi-simplicial sets $X^{\mathrm{nd}} \to Y^{\mathrm{nd}}$. □

**Corollary 11.3.10.** *If $X$ is a braced simplicial set, then $\mathrm{sd}(X)$ is braced as well. In particular, by Theorem 11.3.7, we have canonical homeomorphisms*

$$|X| \simeq |\mathrm{sd}^n(X)|$$

*for all $n \geq 0$.*

*Proof.* By Corollary 11.2.16, $\mathrm{sd}(X)$ is still braced, so we can apply Theorem 11.3.7 repeatedly. □

## 11.4 Simplicial approximation for posets

In this section, we prove the following result, which is a key technical step in our ultimate simplicial approximation theorem.

**Proposition 11.4.1.** *Let $P, Q$ be two posets, with $P$ finite. For any continuous map*

$$f : |N(P)| \to |N(Q)|,$$

*there exists some $n \gg 0$ and some monotone map $\varphi : \mathrm{sd}^n(P) \to Q$ such that the composite map*

$$|N(P)| \simeq |N(\mathrm{sd}^n(P))| \xrightarrow{|N(\varphi)|} |N(Q)|$$

*is homotopic to $f$, where the first homeomorphism is the one from Corollary 11.3.10.*

### 11.4.1 On nerves of posets

To prove Proposition 11.4.1, we start by describing the geometric realization of the nerve of a poset directly. There is fortunately a rather simple construction.

**Ideation 11.4.2.** Let $P$ be a poset. Then simplices of $N(P)$ are determined by their vertices. As we can recognize a simplex purely from its vertices, it is tempting to try and embed $|N(P)|$ in $\mathbb{R}^P$; use the support of a point to determine which simplex it corresponds to, and then use the coordinates to figure out where in the corresponding topological simplex it is.

**Construction 11.4.3.** Let $P$ be a finite poset. Define

$$\varphi_P : |N(P)| \to \mathbb{R}^P$$

the following continuous function: on the non-degenerate $n$-simplex of $|N(P)|$ corresponding to the chain $p_0 < p_1 < \cdots < p_n$ of $P$, we let $\varphi(t_0, ..., t_n)$ be the point whose $p_i$ coordinate is $t_i$, and all of whose other coordinates are zero. It is easy to see this is well-defined and continuous.

**Lemma 11.4.4.** *The image of the continuous function $\varphi_P$ of Construction 11.4.3 is the subset $S_P \subseteq \mathbb{R}^P$ of points $\{x_p\}_{p \in P}$ such that*

1. $\sum_{p \in P} x_p = 1$,
2. $x_p \in [0, 1]$ *for all* $p$,
3. *the set $C = \{p \in P \mid x_p \neq 0\}$ forms a non-empty chain.*

*The map $\varphi_P$ is a homeomorphism onto its image.*

*Proof.* The claimed characterization of the image is obvious. As $P$ is finite, $|N(P)|$ is compact Hausdorff, so to conclude we need only prove that $\varphi_P$ is injective; but this is obvious. □

This section's goal (Proposition 11.4.1) involves a subdivision, though. Thus we also need to understand how this $S_P$ representation of $|N(P)|$ changes when we subdivide.

**Construction 11.4.5.** Let $P$ be a finite poset. For $\sigma$ a simplex of $|N(P)|$, write $b_\sigma \in |N(P)|$ for its barycenter.

We define a continuous function

$$\psi_P : S_{\mathrm{sd}(P)} \to \mathbb{R}^P$$

by

$$\psi_P((x_\sigma)_{\sigma \in \mathrm{sd}(P)}) = \sum_{\sigma \in \mathrm{supp}(x)} x_\sigma b_\sigma.$$

**Proposition 11.4.6.** *Let $P$ be a finite poset. Then the continuous function $\psi_P$ of Construction 11.4.5 has image $S_P$, and the diagram*

$$\begin{array}{ccc} |N(\mathrm{sd}(P))| & \xrightarrow{\alpha} & |N(P)| \\ \downarrow{\scriptstyle \varphi_{\mathrm{sd}(P)}} & & \downarrow{\scriptstyle \varphi_P} \\ S_{\mathrm{sd}(P)} & \xrightarrow{\psi_P} & S_P \end{array}$$

*commutes, where the top horizontal map is the homeomorphism of Corollary 11.3.10, and the vertical maps are the homeomorphisms of Lemma 11.4.4.*

*Proof.* Take $x \in |N(\mathrm{sd}(P))|$. Then $x$ lies in the interior of a unique simplex of $\mathrm{sd}(P)$ (viewing the interior of a 0-dimensional simplex as the sole point of that simplex); call this simplex

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_k,$$

where $\sigma_i \in \mathrm{sd}(P)$. Given this information, $x$ is the data of a tuple $(t_0, ..., t_k)$ of *positive* reals summing to 1.

To calculate $\alpha(x)$, we need to use the map

$$|B^{|\sigma_k|-1}| \to \Delta^{|\sigma_k|-1}.$$

Using this, we see that

$$\alpha(x) = \sum_{i=0}^{k} t_i b_{\sigma_i}.$$

Thus our diagram commutes, as desired. □

In our arguments, it will be helpful to use $S_P$ as our fixed model of $|N(P)|$; we then think of subdivisions as just geometrically subdividing $S_P$.

### 11.4.2 Creating homotopies

**Ideation 11.4.7.** Recall our end goal: we are trying to prove that any continuous map $f : |N(P)| \to |N(Q)|$ is homotopic to some map arising from geometric realization. In particular, at some point in this argument we're going to need to construct a homotopy.

With this goal in mind, viewing $|N(Q)|$ as a subset of $\mathbb{R}^Q$ gives us a great tool for making homotopies between maps $f, g : |N(P)| \to |N(Q)|$: straight line homotopies.

Indeed, we can hope to define

$$H(t, x) := (1 - t)f(x) + tg(x)$$

to get a homotopy from $f$ to $g$; as long as the line from $f(x)$ to $g(x)$ lies in $S_Q$, this will be a valid homotopy.

The set $S_Q$ is built out of convex subsets (namely the simplices of $|N(Q)|$), and so as long as we can prove that $f(x), g(x)$ always lie in the same simplex, then this straight line homotopy trick will work.

**Lemma 11.4.8** (Convex decomposition lemma)**.** *Fix $X$ a compact Hausdorff space and $P$ a finite poset. Let $f, g : X \to |N(P)|$ be two continuous functions.*

*Assume, for every $x \in X$, that we can find some simplex $\tau_x$ of $|N(P)|$ containing both $f(x)$ and $g(x)$. Then $f$ and $g$ are homotopic.*

*Proof.* Use a straight-line homotopy; by the convexity assumption, $tf(x) + (1 - t)g(x)$ always lies in $S_P$. □

**Ideation 11.4.9.** Take $f, g : |N(P)| \to |N(Q)|$ two continuous functions. Suppose we want to apply Lemma 11.4.8 to show $f \simeq g$.

Then, we need it to be the case that, for every $x \in |N(P)|$, the set $\operatorname{supp} f(x) \cup \operatorname{supp} g(x) \subseteq Q$ is a chain. Let's change the framing a little bit; in our target theorem, $f$ is given and we must construct $g$.

Thus we need to choose $g$ such that $g(x), f(x)$ always lie in a common simplex.

As simplices are closed, a single point might lie in multiple simplices. The smallest simplex containing $f(x)$ is the simplex $\sigma$ of $N(Q)$ determined by the chain $\operatorname{supp} f(x)$. However, any simplex $\tau$ of $N(Q)$ with $\sigma$ as a face also contains $f(x)$; this inspires the following notation.

**Notation 11.4.10.** For $P$ a poset, and $\sigma \in \operatorname{sd}(P)$ a chain, we let $C(\sigma) := \bigcup_{\tau \supseteq \sigma} |\tau|$, viewed as a subset of $|N(P)|$.

### 11.4.3 Constructing our monotone map

**Ideation 11.4.11.** Let's ignore the subdivisions for now. Suppose we want to create a monotone map $\varphi : P \to Q$ such that $g = |N(\varphi)|$ is homotopic to $f$, and that we want to produce the homotopy using Lemma 11.4.8. What can we do?

As explained in Ideation 11.4.9, for Lemma 11.4.8 to apply, we need it to be the case that

$$g(x) \in C(\operatorname{supp} f(x))$$

for all $x \in |N(P)|$.

We control $g(p)$ for points $p \in P$ quite directly; indeed, $g(p)$ is just the point all of whose coordinates are zero, except the coordinate in position $\varphi(p)$, which is 1. Thus we need to choose $\varphi$ such that $\varphi(p)$ is always comparable to $\operatorname{supp} f(p)$.

From this perspective, it is natural to try and choose $\varphi$ such that $\varphi(p) \in \operatorname{supp} f(p)$ for all $p$; this is the easiest way to get the comparability. Which element of $\operatorname{supp} f(p)$, though? Probably either max or min; let's try seeing if either of those choices results in a monotone $\varphi$.

Thus we're asking: if $p \leq p'$, how are $\operatorname{supp} f(p)$ and $\operatorname{supp} f(p')$ related? And the answer, unfortunately, is that $f$ was some arbitrary continuous map and so there's no way to get a link.

However, we do know that a subdivision is needed; and when we subdivide $P$, the poset $\operatorname{sd}(P)$ has elements the simplices of $N(P)$, with the order relation being inclusion. Inclusion *is* a geometric condition which is seen by our continuous map $f$. So, can we create a monotone map

$$\varphi : \operatorname{sd}(P) \to Q$$

with $g = |N(\varphi)|$ homotopic to $f$?

Let's try constructing $\varphi(\sigma)$ as we did above. Start with $f(\sigma)$, which is now some compact subset of $|N(Q)|$. Above, our next step was to consider $\operatorname{supp} f(p)$. This is slightly complicated with $f(\sigma)$, as $\operatorname{supp} f(\sigma)$ no longer makes sense. However, we can ask: what elements of $Q$ are comparable to the support of every element of $f(\sigma)$?

For this, it is helpful to introduce the set of points in $|N(Q)|$ which are comparable to some $q$. This leads to the following notion.

**Notation 11.4.12.** For $P$ a finite poset, and $p \in P$, we let $\operatorname{St}^\circ(p) \subseteq |N(P)|$ denote the *open star* at $p$: the subset of points $x \in |N(P)|$ such that $x_p \neq 0$.

The *star* of $p$, denoted $\operatorname{St}(p)$, is the closure of $\operatorname{St}^\circ(p)$.

**Remark 11.4.13.** We call these sets stars because, for any $x \in \operatorname{St}(p)$, the line from $x$ to $p$ lies in $\operatorname{St}(p)$.

**Remark 11.4.14.** The star at $p$ is easily seen to be the union of all simplices in $|N(P)|$ containing $p$ as a vertex.

**Ideation 11.4.15.** Let's continue Ideation 11.4.11. If we'd like $\varphi(\sigma)$ to be some $q$ such that $q$ is comparable to every element in the support of a point of $f(\sigma)$, then we are asking that

$$f(\sigma) \subseteq \operatorname{St}(q).$$

Equivalently, we'd like $\sigma \subseteq f^{-1}(\operatorname{St}(q))$. We can then try to define

$$\varphi(\sigma) := \min\{q \in Q \mid \sigma \subseteq f^{-1}(\operatorname{St}(q))\}.$$

Let's ignore the existence of this minimum for now; the reason we use it is because if $\sigma \subseteq \tau$ and $\tau \subseteq f^{-1}(\operatorname{St}(q))$, then of course $\sigma \subseteq f^{-1}(\operatorname{St}(q))$, so that $\varphi(\sigma) \leq \varphi(\tau)$.

To get this minimum to exist, we need the set we're taking the minimum of to be a non-empty chain. Non-emptiness is actually quite easy to achieve: just subdivide $|N(P)|$ many times until all the simplices lie in some $f^{-1}(\operatorname{St}(q))$. To understand if all the elements in our set are comparable, assume

$$f(\sigma) \subseteq \operatorname{St}(q) \cap \operatorname{St}(q').$$

Then in particular $\mathrm{St}(q), \mathrm{St}(q')$ contain a common point; the support of that point is comparable to both $q$ and $q'$ individually, but unfortunately does not need to contain either $q$ or $q'$ in its support, so there's not much we can say.

To fix this, we just use open stars instead of closed stars in our work above. This gives us a strategy to proving our simplicial approximation result; to implement it, we just need to make precise this step about subdivision.

Unfortunately, even when using

$$\varphi(\sigma) = \min\{q \in Q \mid f(\sigma) \subseteq \mathrm{St}^\circ(q)\},$$

there's still a problem.

A general point $x \in |N(P)|$ lies in the interior of a unique simplex $\sigma$. We may then write

$$x = \sum_{i=0}^{m} \lambda_i b_{\tau_i}$$

where each $\lambda_i > 0$ and the $b_{\tau_i}$ are barycenters of simplices $\tau_0 \subsetneq \cdots \subsetneq \tau_m \subseteq \sigma$. Then

$$g(x) = \sum_{i=0}^{m} \lambda_i \varphi(\tau_i),$$

so that to prove $g(x), f(x)$ are contained in a common simplex, we need only prove that every $\varphi(\tau_i)$ is comparable to $\operatorname{supp} f(x)$.

What do we know about $f(x)$? Well, that $f(x) \in f(\sigma) \subseteq \mathrm{St}^\circ(\varphi(\sigma))$. But this is, unfortunately, not quite what we need to know: we really need to know

$$f(\sigma) \subseteq \bigcap_{i=0}^{m} \mathrm{St}^\circ(\varphi(\tau_i)).$$

This suggests we define

$$\varphi(\sigma) := \max\{q \in Q \mid f(C(\sigma)) \subseteq \mathrm{St}^\circ(q)\}.$$

Here we switched from min to max because if $\sigma \subseteq \tau$ then $C(\tau) \subseteq C(\sigma)$. Recall that $C(\sigma)$ was defined in Notation 11.4.10.

**Lemma 11.4.16.** *Let $\epsilon > 0$. There is some $k \gg 0$ such that, under the homeomorphism $|N(\mathrm{sd}^k P)| \simeq |N(P)|$, the image of every simplex of $|N(\mathrm{sd}^k P)|$ in $S_P$ has diameter at most $\epsilon$.*

*Proof.* Set $d := \dim N(P)$, and let $M_k$ be the maximum diameter of a simplex after $k$ subdivisions (measuring distances in $S_P$).

When $d = 0$, the theorem is trivial (and in fact we can take $k = 1$). Thus assume $d \geq 1$.

Let $\sigma$ be a simplex in $\mathrm{sd}^{k-1}(P)$. When we subdivide another time, $\sigma$ is split into pieces; how big are those pieces?

As a simplex is convex, its diameter is the length of its longest edge. Edges of simplices in the subdivision of simplices arise as follows: take $\tau' \subsetneq \tau \subseteq \sigma$ two faces of $\sigma$, and connect their barycenters

$$b_\tau = \frac{1}{|\tau|} \sum_{x \in \tau} x,$$

$$b_{\tau'} = \frac{1}{|\tau'|} \sum_{x \in \tau'} x.$$

As $\tau' \subsetneq \tau$, we may write

$$b_\tau - b_{\tau'} = \frac{|\tau'| - |\tau|}{|\tau|} b_{\tau'} + \frac{1}{|\tau|} \sum_{x \in \tau \setminus \tau'} x = \frac{|\tau'| - |\tau|}{|\tau|} b_{\tau'} + \frac{|\tau| - |\tau'|}{|\tau|} b_{\tau \setminus \tau'}.$$

Thus

$$d(b_\tau, b_{\tau'}) \leq \frac{|\tau \setminus \tau'|}{|\tau|} d(b_{\tau'}, b_{\tau \setminus \tau'}) \leq \frac{d}{d+1} M_{k-1}.$$

In particular,

$$M_k \leq \left(1 - \frac{1}{d+1}\right) M_{k-1},$$

which of course implies our desired result by letting $k \to \infty$. □

By Lemma 11.4.16 and the Lebesgue number lemma, we can choose $k \gg 0$ so that each $C(\sigma)$ for a simplex $\sigma$ of $|N(\mathrm{sd}^k P)|$ is mapped via $f$ into some open star $\mathrm{St}^\circ(q)$. Set $P' := \mathrm{sd}^k P$.

**Construction 11.4.17.** We define a monotone map

$$\varphi : \mathrm{sd}(P') \to Q$$

sending $\sigma$ (viewed as a simplex of $P'$) to the largest $q$ such that $f(C(\sigma)) \subseteq \mathrm{St}^\circ(q)$.

There exists a largest such $q$ because the set

$$A := \{q \in Q \mid f(C(\sigma)) \subseteq \mathrm{St}^\circ(q)\}$$

forms a non-empty chain. The non-emptiness is by our choice of $k$; to see it is a chain, take $q, q' \in A$. Then $f(C(\sigma)) \subseteq \mathrm{St}^\circ(q) \cap \mathrm{St}^\circ(q')$, and in particular this intersection is nonempty; thus there is some point $x$ having both $q, q'$ in its support, implying $q, q'$ are comparable.

The monotonicity is immediate from the observation that, when $\sigma \subseteq \tau$, we have $C(\tau) \subseteq C(\sigma)$.

**Lemma 11.4.18.** *The continuous functions $|N(\varphi)|$ and $f$ are homotopic.*

*Proof.* Set $g := |N(\varphi)|$. By Lemma 11.4.8, it suffices to prove that, for every $x \in |N(P')|$, the points $f(x), g(x)$ lie in the same simplex of $|N(Q)|$.

A general point $x \in |N(P')|$ lies in the interior of a unique simplex $\sigma$ of $|N(P')|$. Write the vertices of this simplex as

$$p'_0 < p'_1 < \cdots < p'_n,$$

so that

$$x = \sum_{i=0}^{n} t_i p'_i$$

with each $t_i > 0$ and with $1 = \sum_i t_i$. We can rewrite this point $x$ in the form

$$x = \sum_{i=0}^{m} \lambda_i b_{\tau_i}$$

for some simplices $\tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_m \subseteq \sigma$. Thus

$$g(x) = \sum_i \lambda_i \varphi(\tau_i).$$

Thus

$$\operatorname{supp} g(x) \subseteq \{\varphi(\tau_0), ..., \varphi(\tau_m)\},$$

and so we need only prove each $\varphi(\tau_i)$ is comparable to $\operatorname{supp} f(x)$. And fortunately for us, as $x \in \sigma$, we have $x \in C(\tau_i)$ for each $i$, and so

$$f(x) \in f(C(\tau_i)) \subseteq \mathrm{St}^\circ(\varphi(\tau_i)),$$

immediately implying our desired comparability. □

*Proof of Proposition 11.4.1.* When $Q$ is finite, apply Lemma 11.4.18. But in the general case, the image of $f$ is compact and hence is contained in $|N(Q')|$ for some finite subset $Q'$ of $Q$, and we can then apply Lemma 11.4.18 to $Q'$. □

## 11.5 The last vertex map

**Ideation 11.5.1.** In the last section, we proved simplicial approximation for nerves of posets. We would like to make this work for arbitrary simplicial sets, but at present there is a big challenge to implementing this: to even state simplicial approximation, we need the homeomorphism $|\operatorname{sd}^n(X)| \simeq |X|$, which in general only exists for braced simplicial sets.

However, we can still reasonably interpret simplicial approximation if we ask for a canonical *homotopy equivalence* $|\operatorname{sd}(X)| \to |X|$ for every simplicial set $X$. In this section, we construct such homotopy equivalences. And these are actually much more convenient than the homeomorphisms for braced simplicial sets, because these homotopy equivalences end up coming as geometric realizations of simplicial set maps $\operatorname{sd}(X) \to X$.

With Proposition 8.6.1 in mind, we will construct a weak equivalence $\lambda_X : \operatorname{sd}(X) \to X$.

**Ideation 11.5.2.** Let $X = N(P)$ be the nerve of a poset; then $\operatorname{sd}(X) = N(\operatorname{sd}(P))$ is pretty simple. Is there some natural, easy-to-define map

$$\operatorname{sd}(X) \to X$$

which is a candidate for our weak equivalence?

The easiest way to create a map $\operatorname{sd}(X) \to X$ would be to create a monotone map $\operatorname{sd}(P) \to P$. A point of $\operatorname{sd}(P)$ is a chain $C = p_0 < \cdots < p_n$ in $P$. There is thus a natural monotone map

$$\max : \operatorname{sd}(P) \to P$$

sending a chain to its maximal element; because chains in $\operatorname{sd}(P)$ are ordered by inclusion, max is monotone (observe that min would not be monotone in general!).

What does $N(\max)$ look like on geometric realizations? Take $X = \Delta_1$, so $P = [1]$. Then $\operatorname{sd}(P)$ is the poset $\{\{0\}, \{1\}, \{0,1\}\}$ (ordering by inclusion), and $N(\operatorname{sd}(P))$ is the 1-dimensional simplicial set whose 0-simplices are the points of $\operatorname{sd}(P)$, and having two non-degenerate 1-simplices

$$\{0\} \subseteq \{0,1\},$$

$$\{1\} \subseteq \{0,1\}.$$

In other words, $N(\mathrm{sd}(P))$ has three vertices $a, a', b$, together with two edges $\ell, \ell'$ going, respectively, from $a$ to $b$ and from $a'$ to $b$.

The map $N(\mathrm{sd}(P)) \to \Delta_1$ sends $a$ to 0, $a'$ to 1, and $b$ to 1; it sends $\ell$ to the unique non-degenerate 1-simplex of $\Delta_1$, and it sends $\ell'$ to the degenerate 1-simplex at the vertex 1.

In particular, $|N(\max)|$ is not a homeomorphism; however, it is a homotopy equivalence, contracting down the extra edge of $N(\mathrm{sd}(P))$. This seems promising, so we'll go with it.

**Construction 11.5.3** (The last vertex map)**.** Let $X$ be a simplicial set. We define the *last vertex map*

$$\lambda_X : \mathrm{sd}(X) \to X$$

as follows.

An $n$-simplex of $\mathrm{sd}(X)$ can be written in canonical form as a triple $(m, \sigma \in X_m, S_\bullet)$, with $S_0 \subseteq \cdots \subseteq S_n = [m]$ a chain of sets and $\sigma$ non-degenerate.

Observe that the chain

$$\max S_0 \leq \max S_1 \leq \cdots \leq \max S_n$$

defines a monotone map $\varphi : [n] \to [m]$; we define

$$\lambda_X(m, \sigma, S_\bullet) := \varphi^* \sigma.$$

**Theorem 11.5.4.** *Let $X$ be a simplicial set. Then the last vertex map $\lambda_X$ is a weak equivalence.*

The proof of Theorem 11.5.4 will occupy the rest of this section.

**Ideation 11.5.5.** By Theorem 8.5.4, it suffices to prove Theorem 11.5.4 assuming $X$ is $n$-dimensional for some $n$, as we can then take a filtered colimit to deduce the general case. The advantage to doing this is that we can *induct on $n$*.

So, assume $X$ is $n$-dimensional. Set $Y = \mathrm{sk}_{n-1}(X)$. Write $X$ as a pushout

$$\begin{array}{ccc} \coprod_{\sigma \in S} \partial\Delta_n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in S} \Delta_n & \longrightarrow & X. \end{array}$$

Fix a Kan complex $Q$. We must prove that the map

$$\theta : [X, Q] \to [\mathrm{sd}(X), Q]$$

is a homotopy equivalence.

To prove this is a homotopy equivalence, we use that $X$ is a pushout to write

$$[X, Q] = [Y, Q] \times_{[\coprod_\sigma \partial\Delta_n, Q]} [\coprod_\sigma \Delta_n, Q].$$

The inductive hypothesis will help us understand the $[Y, Q]$ term; the $[\Delta_n, Q]$ term we must study directly.

**Lemma 11.5.6.** *The last vertex map $\lambda_{\Delta_n}$ of an $n$-simplex is a weak equivalence.*

*Proof.* By Theorem 8.3.2, the simplicial set $\Delta_n$ is homotopy equivalent to a point. But $\mathrm{sd}(\Delta_n)$ is defined as the nerve of a poset having a largest element, so by the proof of Theorem 8.3.2, $\mathrm{sd}(\Delta_n)$ is also homotopy equivalent to a point. By the 2-out-of-3 theorem, it follows that any map between $\Delta_n$ and $\mathrm{sd}(\Delta_n)$ is a weak equivalence. □

*Proof of Theorem 11.5.4.* We have a commutative diagram

$$\begin{array}{ccc} [X,Q] & \longrightarrow & [\mathrm{sd}(X),Q] \\ \downarrow & & \downarrow \\ [Y,Q] & \longrightarrow & [\mathrm{sd}(Y),Q], \end{array}$$

where the bottom map is a homotopy equivalence by our induction hypothesis, and the vertical maps are both Kan fibrations (as $Y \hookrightarrow X$ is an inclusion, so is $\mathrm{sd}(Y) \hookrightarrow \mathrm{sd}(X)$). By Theorem 9.5.3, to prove that the top arrow is an equivalence, it suffices to prove that, for any $f : Y \to Q$, the map

$$\theta_f : \{f\} \times_{[Y,Q]} [X,Q] \to \{f\lambda_Y\} \times_{[\mathrm{sd}(Y),Q]} [\mathrm{sd}(X),Q]$$

is a homotopy equivalence of Kan complexes.

As $X$ is given by a certain pushout, giving a map $X \to Q$ extending $f : Y \to Q$ is the same as giving a map $\tilde{T} := \coprod_{\sigma \in S} \Delta_n \to Q$ whose restriction to $T := \coprod_{\sigma \in S} \partial\Delta_n$ is some fixed function $f'$. In other words, we can rewrite $\theta_f$ as a map

$$\theta_{f'} : \{f'\} \times_{[T,Q]} [\tilde{T},Q] \to \{f'\lambda_T\} \times_{[\mathrm{sd}(T),Q]} [\mathrm{sd}(\tilde{T}),Q].$$

Here, we use that sd is defined as a geometric realization, and hence commutes with arbitrary colimits.

Using the other direction of Theorem 9.5.3, to check that $\theta_{f'}$ is a homotopy equivalence, it suffices to prove that

$$[\tilde{T},Q] \to [\mathrm{sd}(\tilde{T}),Q]$$

and

$$[T,Q] \to [\mathrm{sd}(T),Q]$$

are homotopy equivalences. As homotopy equivalences are stable under arbitrary products (just multiply the homotopy inverses together!), the first homotopy equivalence follows from Lemma 11.5.6; for the second, use the induction hypothesis. Thus $\theta_{f'}$ is a homotopy equivalence, so $\theta_f$ is, and Theorem 11.5.4 is proven! □

### 11.5.1 Anodyne applications

As one fun application, we remark that sd also sends weak equivalences to weak equivalences.

**Corollary 11.5.7.** *If $f : X \to Y$ is a weak equivalence, then $\mathrm{sd}(f)$ is a weak equivalence.*

*Proof.* As the last vertex map is a natural transformation $\mathrm{sd} \to \mathrm{id}$, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{sd}(X) & \xrightarrow{\mathrm{sd}(f)} & \mathrm{sd}(Y) \\ \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \lambda_Y} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

As $f \circ \lambda_X$ is a composition of weak equivalences, it is a weak equivalence; therefore (using the 2-out-of-3 rule) we deduce that $\mathrm{sd}(f)$ is a weak equivalence as well. □

The subdivision functor preserves both weak equivalences and inclusions; this suggests it should preserve strongly anodyne morphisms. Fortunately for us, it does.

**Proposition 11.5.8.** *If $i : A \hookrightarrow B$ is strongly anodyne, then so is* $\mathrm{sd}(i)$.

*Proof.* This proof follows the strategy of Kerodon tag 00Z3. The functor sd is a left adjoint functor; therefore it sends isomorphisms to isomorphisms, compositions to compositions, $\omega$-compositions to $\omega$-compositions, pushouts and retracts to pushouts and retracts, and arbitrary coproducts of morphisms to coproducts.

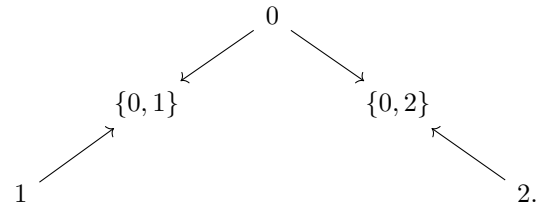
Thus to prove sd sends strongly anodyne maps to strongly anodyne maps, it suffices to prove that sd sends horn inclusions to strongly anodyne maps. This is fortunately pretty easy because $\Lambda^i_n$ and $\Delta_n$ are both braced, so they have easy to describe subdivisions.

From the definition, $\mathrm{sd}(\Delta_n) = N(C_n)$, for $C_n$ the poset of nonempty chains in $[n]$. The category $P^{\mathrm{nd}}\Lambda^i_n$ is actually the poset $C^i_n := C_n \setminus \{[n], [n] \setminus \{i\}\}$. Thus the map

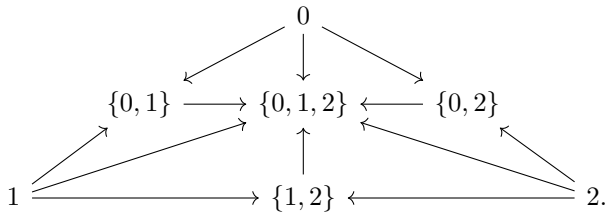
$$\mathrm{sd}(\Lambda^i_n) \to \mathrm{sd}(\Delta_n)$$

is the nerve of the inclusion of posets $C^i_n \hookrightarrow C_n$, as our identification $N(P^{\mathrm{nd}}X) \simeq \mathrm{sd}(X)$ for braced $X$ was functorial in morphisms sending non-degenerate simplices to non-degenerate simplices.

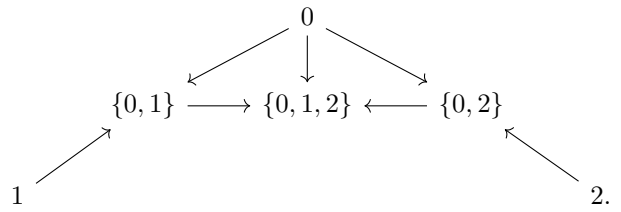
It is helpful to draw an example situation, and remember that we can always build strongly anodyne morphisms by attaching simplices along horn inclusions. The subdivision $\mathrm{sd}(\Lambda^0_2)$ looks as follows.




Recall that $\mathrm{sd}(\Delta_2)$, the shape we need to build by attaching simplices along horns, is




To begin building, it seems natural to attach the 1-simplex $0 \to \{0, 1, 2\}$, and then to attach the two 2-simplices involving 0 and $\{0, 1, 2\}$ along the resulting horns; this produces the simplicial set




In general, this suggests we start by attaching all the simplices of $\mathrm{sd}(\Delta_n)$ which touch $i$. This can be accomplished in one big step: set $P$ the lattice of *all* subsets of $[n] \setminus \{i\}$ (including the emptyset) and consider the pushout

$$\begin{array}{ccc} N(P \setminus \{[n] \setminus \{i\}\}) & \xrightarrow{S \mapsto S \cup \{i\}} & \mathrm{sd}(\Lambda^i_n) \\ \downarrow & & \downarrow \\ N(P) & \longrightarrow & Z, \end{array}$$

where the unioning with $i$ is motivated by our desire to force all the simplices to include $i$. To prove this inclusion $\mathrm{sd}(\Lambda^i_n) \hookrightarrow Z$ is strongly anodyne, it suffices to check that $N(P_-) \hookrightarrow N(P)$ (for $P_- := P \setminus \{[n] \setminus \{i\}\}$) is strongly anodyne. And fortunately, this is easy: the inclusion $N(P_-) \hookrightarrow N(P)$ is obtained by taking the $n$-fold Leibniz product of $\{0\} \hookrightarrow \Delta_1$ (this is most easily seen by induction on $n$).

Finally, we need to attach the remaining simplices. In our $\Lambda^0_2$ example, from $Z$ it seems we should first add the vertex $\{1, 2\}$, then the 1-simplices involving $\{1, 2\}$, and then the 2-simplices. Let's consider briefly the new simplices we've added, to see if this 'new' simplicial set is easy to describe.

Our attachments are all going to involve simplices in the bottom part of our diagram; these are the simplices $\{0, 1\}, \{0, 1, 2\}, \{0, 2\}$, and 1 and 2. This suggests that, when generalizing, we look at simplices which either do not involve $i$ at all, or which contain $i$ and something else.

Define $P_+ = P \setminus \{\emptyset\}$, so that we can phrase the simplices we keep as belonging to $N(P_+)$ (again with the attaching map $S \mapsto S \cup \{i\}$), or to the image of $N(P_\pm) \hookrightarrow Z$ where $P_\pm = P_+ \cap P_-$ and this inclusion is now the 'obvious' one $S \mapsto S$.

With the goal in mind of 'try to make this attaching map be strongly anodyne,' it is instructive to ask if we can write the union of $N(P_+)$ and $N(P_\pm)$ as some kind of Leibniz product. Let's define

$$h : \Delta_1 \times N(P_+) \to \mathrm{sd}(\Delta_n)$$

as the nerve of the map of posets

$$(0, S) \mapsto S, (1, S) \mapsto S \cup \{i\}.$$

Then we have a pushout diagram

$$\begin{array}{ccc} (\{1\} \times N(P_+)) \coprod_{\{1\} \times N(P_\pm)} (\Delta_1 \times N(P_\pm)) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \Delta_1 \times N(P_+) & \longrightarrow & \mathrm{sd}(\Delta_n), \end{array}$$

so that $Z \hookrightarrow \mathrm{sd}(\Delta_n)$ is strongly anodyne. We conclude. □

**Corollary 11.5.9.** *If $f : X \to S$ is a Kan fibration, then so is* $\mathrm{Ex}(f)$.

*Proof.* Immediate from Proposition 11.5.8 by adjunction. □

## 11.6 Simplicial approximation for braced simplicial sets

We can now prove simplicial approximation for braced simplicial sets. The idea is to reduce to the case of the nerve of a poset.

**Ideation 11.6.1** (Simplifying $S$)**.** For a braced simplicial set $S$, by Theorem 11.2.15 we have $\mathrm{sd}(S) \simeq N(P^{\mathrm{nd}}S)$.

The nerve of a category is a little simpler than an arbitrary simplicial set, so replacing $S$ by $\mathrm{sd}(S)$ makes our life a little easier.

However, $\mathrm{sd}^2(S)$ is even simpler, because the category $P^{\mathrm{nd}}(N(P^{\mathrm{nd}}S))$ is actually even a poset: that is, all the faces of a non-degenerate simplex of $N(P^{\mathrm{nd}}S)$ are distinct. To see this, just observe that a non-degenerate simplex of $N(P^{\mathrm{nd}}S)$ is a chain

$$\sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_0$$

of non-degenerate simplices in $S$, where $\sigma_i$ is a (proper) face of $\sigma_{i-1}$. In particular, all the simplices in this chain are distinct, so that all different boundaries (which are calculated by deleting terms from this chain) are distinct.

Thus $\mathrm{sd}^2(S)$ is the nerve of a *poset*.

**Theorem 11.6.2** (Simplicial approximation for braced simplicial sets)**.** *Let $S, T$ be two* braced *simplicial sets, with $S$ finite. Let $f : |S| \to |T|$ be continuous. Then there exists some $n \gg 0$ and some simplicial set map $\varphi : \mathrm{sd}^n(S) \to T$ such that the composite map*

$$|S| \simeq |\,\mathrm{sd}^n(S)| \xrightarrow{|\varphi|} |T|$$

*is homotopic to $f$, where the homeomorphism $|S| \simeq |\,\mathrm{sd}^n(S)|$ used in the above composition is the one constructed in Corollary 11.3.10.*

*Proof.* Replacing $S$ with $\mathrm{sd}^2(S)$, we can assume that $S$ is the nerve of a poset; as $S$ is finite, this poset will also be finite.

Similarly, $\mathrm{sd}^2(T)$ is the nerve of a poset, and the last vertex map

$$\lambda^2_T : \mathrm{sd}^2(T) \to T$$

is a weak equivalence. Thus $|\lambda^2_T|$ is a homotopy equivalence, so choose $\psi : |T| \to |\,\mathrm{sd}^2(T)|$ a homotopy inverse. To find a simplicial approximation of $f$ by a map $|\varphi|$ as desired, it suffices to create a map $\varphi' : \mathrm{sd}^n(S) \to \mathrm{sd}^2(T)$ whose geometric realization is homotopic to $\psi \circ f$ to conclude, since we can then compose $\varphi'$ with the last vertex map.

But after these reductions, both $S$ and $\mathrm{sd}^2(T)$ are nerves of posets, and so we may apply Proposition 11.4.1 to conclude. □

# Chapter 12

# Kan's $\mathrm{Ex}^\infty$-functor

## 12.1 Introduction

In the previous chapter, we used the geometry of barycentric subdivision to prove a simplicial approximation theorem for *braced* simplicial sets. To go all the way to arbitrary simplicial sets requires a bit more algebraic work.

Recall that, in simplicial approximation, we are attempting to create some morphism of simplicial sets $\mathrm{sd}^n(S) \to T$. One of the annoying things in working with simplicial approximation is that we do not know what $n$ should be. For this reason, it is natural to consider the limit

$$\mathrm{sd}^\infty(S) := \lim(\cdots \to \mathrm{sd}^2(S) \to \mathrm{sd}(S) \to S),$$

where the transition maps are always the last vertex map, and attempt to create a single map $\mathrm{sd}^\infty(S) \to T$. Unfortunately, here we lose some control: we don't know if this map will come from some finite stage $\mathrm{sd}^n(S)$ of the construction or not.

However, Kan had a really clever algebraic trick. Let $\rho_T : T \to \mathrm{Ex}(T)$ denote the adjoint to the last vertex map, where recall from last chapter that sd admits a right adjoint Ex . Then we can define

$$\mathrm{Ex}^\infty(T) := \mathrm{colim}(T \to \mathrm{Ex}(T) \to \mathrm{Ex}^2(T) \to \cdots),$$

and search for functions $S \to \mathrm{Ex}^\infty(T)$ instead; when $S$ is finite, any such map will factor through some finite stage of the colimit by a compactness argument, and so we'll get a map $S \to \mathrm{Ex}^n(T)$, and hence a map $\mathrm{sd}^n(S) \to T$.

The aforementioned compactness argument is quite simple, so we give it now.

**Proposition 12.1.1.** *Let $X$ be a finite simplicial set, and consider a* filtered *diagram $\{Y_i\}_{i\in I}$ in* sSet . *Any map*

$$X \to \mathrm{colim}_i\, Y_i$$

*factors through some $Y_i$.*

**Remark 12.1.2.** This is a general categorical principle, that appropriately 'finite' objects always factor through some finite stage of a filtered colimit. The notion of *$\kappa$-compact objects* of a category makes this precise.

*Proof.* Suppose $X$ has no non-degenerate simplices of dimension exceeding $d$. Let $N_j$ be the (finite!) set of non-degenerate simplices of $X$ of dimension $j$.

A map of simplicial sets $X \to \operatorname{colim}_i Y_i$ is equivalent to giving a series of maps

$$X_j \to (\operatorname{colim}_i Y_i)_j$$

for each $j \geq 0$, which obey some compatibilities with respect to face and degeneracy operators.

For each $0 \leq j \leq d$, as $N_j$ is finite, and as

$$(\operatorname{colim}_i Y_i)_j = \operatorname{colim}_i (Y_i)_j,$$

with the colimit on the right being computed in Set, there is some index $i_j$ so that the function

$$N_j \to X_j \to (\operatorname{colim}_i Y_i)_j$$

factors through a function $f_j : N_j \to (Y_{i_j})_j$.

Choose some $i' \in I$ dominating each $i_j$. Now, we take care of some relations. For each non-degenerate simplex $\sigma$, and each face $d^k\sigma$, write $d^k\sigma = \pi^*\tau$ with $\pi$ a surjection and $\tau$ non-degenerate. We impose the relation

$$d^k(f_j(\sigma)) = \pi^* f_{\dim \tau}(\tau).$$

There are only finitely many such relations for each $\sigma$, and so filteredness allows us to impose all of them.

As each $N_j$ is finite, we may choose some $i'' \geq i'$ large enough that all the face relations hold. In particular, we have functions $g_j : N_j \to (Y_{i''})_j$ which are compatible with face operators.

Define

$$g : X \to Y_{i''}$$

using the $g_j$ and the skeletal filtration on $X$. More precisely, recall that $X = X(d)$, where $X(0)$ is the 0-dimensional truncation of $X$, and $X(j+1)$ fits into a pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma \in N_{j+1}} \partial\Delta_{j+1} & \longrightarrow & \coprod_{\sigma \in N_{j+1}} \Delta_{j+1} \\ \downarrow & & \downarrow \\ X(j) & \longrightarrow & X(j+1). \end{array}$$

We inductively define maps $X(j) \to Y_{i''}$, using the maps $g_j$ on the non-degenerate simplices, and using that the face relations hold to ensure that the map descends to the pushout.

The given map $X \to \operatorname{colim}_i Y_i$ clearly factors through our constructed map $X \to Y_{i''}$ (as this can be checked by looking at $j$-simplices for all $j$), and so we conclude. □

The downside to this approach is that working with $\mathrm{Ex}^n$ loses some of the geometry behind subdivision; however, it is technically more convenient, and it is hard to work with subdivision geometrically beyond braced simplicial sets anyways, so we shall march ahead with $\mathrm{Ex}^\infty$.

## 12.2 The adjunction preserves homotopies

Key to our approach to simplicial approximation before was that the last vertex map $\lambda_S : \mathrm{sd}(S) \to S$ was a weak equivalence. Similarly, it is true that $\rho_T : T \to \mathrm{Ex}(T)$ is always a weak equivalence. And fortunately for us, it is actually quite easy to prove this from our study of the last vertex map, using the following amazing theorem: the adjunction $(\mathrm{sd}, \mathrm{Ex})$ is compatible with homotopies!

**Theorem 12.2.1.** *Let $X, Y$ be simplicial sets. Assume $Y$ is a Kan complex. Then the bijection*

$$\mathrm{Mor}_{\mathrm{sSet}}(\mathrm{sd}(X), Y) \simeq \mathrm{Mor}_{\mathrm{sSet}}(X, \mathrm{Ex}(Y)) \tag{12.2.2}$$

*respects homotopy: that is, two maps $f, g : \mathrm{sd}(X) \to Y$ are homotopic if, and only if, their adjuncts $f', g' : X \to \mathrm{Ex}(Y)$ are.*

One direction of Theorem 12.2.1 is quite easy.

**Lemma 12.2.3.** *Keep the notation of Theorem 12.2.1. If $f, g$ are homotopic, then $f', g'$ are as well. In fact, this holds even when $Y$ is not Kan.*

*Proof.* Choose a homotopy

$$H : \mathrm{sd}(X) \times \Delta_1 \to Y$$

from $f$ to $g$.

We wish to produce a homotopy from $f'$ to $g'$; as such a homotopy has target $\mathrm{Ex}(Y)$, it is natural to hope we can produce it via adjunction from $H$. Unfortunately, the set $\mathrm{sd}(X) \times \Delta_1$ is not the subdivision of anything, so we can't apply adjunction; however, the composite map

$$\tilde{H} : \mathrm{sd}(X \times \Delta_1) \to \mathrm{sd}(X) \times \mathrm{sd}(\Delta_1) \xrightarrow{\mathrm{id}_{\mathrm{sd}(X)} \times \lambda_{\Delta_1}} \mathrm{sd}(X) \times \Delta_1 \xrightarrow{H} Y$$

is adjoint to a map

$$H' : X \times \Delta_1 \to \mathrm{Ex}(Y),$$

so we can hope that $H'$ is a homotopy from $f'$ to $g'$.

And indeed, $H'|_{X \times \{0\}}$ is adjoint to $\tilde{H}|_{\mathrm{sd}(X \times \{0\})}$, which is just $f$. Similarly, $\tilde{H}|_{\mathrm{sd}(X \times \{1\})} = g$, and so we conclude. □

The reverse direction is a bit harder.

**Ideation 12.2.4.** Suppose $f', g' : X \to \mathrm{Ex}(Y)$ are homotopic, via a homotopy

$$H' : X \times \Delta_1 \to \mathrm{Ex}(Y).$$

Adjunction gives us a map

$$\tilde{H} : \mathrm{sd}(X \times \Delta_1) \to Y,$$

but it's difficult to turn this into a map with domain $\mathrm{sd}(X) \times \Delta_1$.

In our proof of Lemma 12.2.3, we used a certain map

$$\alpha : \mathrm{sd}(X \times \Delta_1) \to \mathrm{sd}(X) \times \Delta_1,$$

which unfortunately goes the wrong way to help us here. However, if $\alpha$ were a *weak equivalence*, then using that $Y$ is a Kan complex, we could hope to turn $\tilde{H}$ into a map

$$H : \mathrm{sd}(X) \times \Delta_1 \to Y.$$

By the 2-out-of-3 rule, since $\alpha$ was defined as a composition of two maps – one of which (the last vertex map) was a weak equivalence – to check $\alpha$ is a weak equivalence, it is equivalent to show that

$$\mathrm{sd}(X \times \Delta_1) \to \mathrm{sd}(X) \times \mathrm{sd}(\Delta_1)$$

is a weak equivalence. Fortunately, this is not so hard to prove.

**Lemma 12.2.5.** *Let $X, Y$ be simplicial sets. Then $\mathrm{sd}(X \times Y) \to \mathrm{sd}(X) \times \mathrm{sd}(Y)$ is a weak equivalence.*

*Proof.* The idea is pretty cute: by the last vertex map, $\mathrm{sd}(X)$ is weakly equivalent to $X$; and of course $X \times Y$ and $X \times Y$ are weakly equivalent! To formalize this, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{sd}(X \times Y) & \longrightarrow & \mathrm{sd}(X) \times \mathrm{sd}(Y) \\ \downarrow & & \downarrow \\ X \times Y & \xrightarrow{\mathrm{id}} & X \times Y, \end{array}$$

where the vertical maps are the last vertex map. By the 2-out-of-3 rule and staring at this diagram, we conclude. □

*Proof of Theorem 12.2.1.* Let $f, g : \mathrm{sd}(X) \to Y$ be two maps, and $f', g' : X \to \mathrm{Ex}(Y)$ be their adjuncts. If $f \simeq g$, then apply Lemma 12.2.3.

Conversely, assume $f' \simeq g'$. Then we have a homotopy $H' : \Delta_1 \times X \to \mathrm{Ex}(Y)$ from $f'$ to $g'$. We wish to construct a homotopy $H : \Delta_1 \times \mathrm{sd}(X) \to Y$, but this is a little trickier. Adjunction on $H'$ gives us a map

$$\tilde{H} : \mathrm{sd}(\Delta_1 \times X) \to Y,$$

but we really need a map with domain $\Delta_1 \times \mathrm{sd}(X)$.

Consider the composition

$$\mathrm{sd}(\Delta_1 \times X) \to \mathrm{sd}(\Delta_1) \times \mathrm{sd}(X) \xrightarrow{\lambda_{\Delta_1} \times \mathrm{id}_{\mathrm{sd}(X)}} \Delta_1 \times \mathrm{sd}(X)$$

from above. As $\lambda_{\Delta_1}$ is a weak equivalence, the map $\mathrm{sd}(\Delta_1) \times \mathrm{sd}(X) \to \Delta_1 \times \mathrm{sd}(X)$ is a weak equivalence as well.

By Lemma 12.2.5 and stability of weak equivalences under products, the composite map

$$\alpha : \mathrm{sd}(\Delta_1 \times X) \to \Delta_1 \times \mathrm{sd}(X)$$

is a weak equivalence.

Thus we can find a map

$$H : \Delta_1 \times \mathrm{sd}(X) \to Y$$

with $H \circ \alpha \simeq \tilde{H}$. In particular, $H|_{\{0\} \times \mathrm{sd}(X)}$ is homotopic to $f$ and $H|_{\{1\} \times \mathrm{sd}(X)}$ is homotopic to $g$, so that (by transitivity of homotopy), the maps $f, g$ are homotopic, as desired. □

As a corollary, we deduce that $\rho_T : T \to \mathrm{Ex}(T)$ is also a weak equivalence. Before proving this, we need one technical lemma.

**Lemma 12.2.6.** *Let $T$ be a simplicial set. Then $\rho_{\mathrm{Ex}(T)}$ and $\mathrm{Ex}(\rho_T)$ are homotopic as maps $\mathrm{Ex}(T) \to \mathrm{Ex}^2(T)$.*

**Remark 12.2.7.** In this proof, we will use a nice trick. The functor Ex is a lot harder to access than sd . However, Theorem 12.2.1 tells us that the homotopy can be checked *after* adjunction, allowing us to turn the Ex's into sd's. Then we may write down a homotopy explicitly in a simple case, and bootstrap.

*Proof.* Applying Lemma 12.2.3 twice, to check that these two maps are homotopic, it suffices to prove that their adjuncts, certain maps

$$\mathrm{sd}^2(\mathrm{Ex}(T)) \to T,$$

are homotopic.

These adjuncts can be described rather concretely; they are $\epsilon_T \circ \mathrm{sd}(\lambda_{\mathrm{Ex}(T)})$ and $\epsilon_T \lambda_{\mathrm{sd}(\mathrm{Ex}(T))}$, where $\epsilon_T : \mathrm{sd}(\mathrm{Ex}(T)) \to T$ is the counit.

Thus we prove in general, for a simplicial set $S$, that $\mathrm{sd}(\lambda_S) \simeq \lambda_{\mathrm{sd}(S)}$, as maps $\mathrm{sd}^2(S) \to \mathrm{sd}(S)$.

When $S = \Delta_n$, this can be done very explicitly, as $\mathrm{sd}(S)$ and $\mathrm{sd}^2(S)$ are both posets. In that case, a vertex of $\mathrm{sd}^2(S)$ is a chain

$$S_0 \subsetneq \cdots \subsetneq S_k$$

of non-empty subsets of $[n]$, and our two maps $\mathrm{sd}(\lambda_S), \lambda_{\mathrm{sd}(S)}$ send this chain to

$$\{\max(S_0), ..., \max(S_k)\},$$

$$S_k,$$

respectively. As the latter set always contains the former, there is a natural transformation $\mathrm{sd}(\lambda_S) \to \lambda_{\mathrm{sd}(S)}$ (viewing these posets as categories), giving us a homotopy between the corresponding simplicial set maps. As this homotopy is natural in $n$, by taking colimits over simplices it extends to arbitrary $S$. □

**Corollary 12.2.8.** *Let $T$ be a simplicial set. Then the adjunct $\rho_T : T \to \mathrm{Ex}(T)$ to the last vertex map $\lambda_T : \mathrm{sd}(T) \to T$ is a weak equivalence.*

*Proof.* Assume first that $T$ is a Kan complex. Then $\rho_T$ is a weak equivalence if and only if, for every simplicial set $S$, the map

$$\pi_0[S, T] \xrightarrow{\rho_T} \pi_0[S, \mathrm{Ex}(T)]$$

is a bijection. But $\pi_0[S, \mathrm{Ex}(T)] = \pi_0[\mathrm{sd}(S), T]$ by adjunction, and the resulting map

$$\pi_0[S, T] \to \pi_0[\mathrm{sd}(S), T] \tag{12.2.9}$$

sends $f : S \to T$ to the composite map

$$\mathrm{sd}(S) \xrightarrow{\mathrm{sd}(f)} \mathrm{sd}(T) \xrightarrow{\lambda_T} T.$$

As the last vertex map is natural in $T$, this composition is equal to $f \circ \lambda_S$. In particular, (12.2.9) is a bijection, because $T$ is a Kan complex and $\lambda_S$ is a weak equivalence.

In the general case, take $Q$ a Kan complex. We wish to show that

$$\pi_0[\mathrm{Ex}(T), Q] \to \pi_0[T, Q] \tag{12.2.10}$$

is a bijection. Choose $r_Q : \mathrm{Ex}(Q) \to Q$ a homotopy inverse to $\rho_Q$, which exists thanks to the proof of this corollary in the Kan case. Then

$$[a] \mapsto [r_Q \circ \mathrm{Ex}(a)]$$

is an inverse to (12.2.10). Indeed, one composite is the identity by naturality of $\rho$; for the other composite, for any $b : \mathrm{Ex}(T) \to Q$ observe that

$$r_Q \circ \mathrm{Ex}(b\rho_T) \simeq r_Q \,\mathrm{Ex}(b)\rho_{\mathrm{Ex}(T)} = r_Q \rho_Q b \simeq b,$$

using Lemma 12.2.6. Thus $\rho_T$ is a weak equivalence, as desired. □

**Corollary 12.2.11.** *Let $T$ be a simplicial set. Then the natural map*

$$T \to \mathrm{Ex}^\infty(T)$$

*is a weak equivalence.*

*Proof.* As a filtered colimit of weak equivalences is a weak equivalence, we conclude from Corollary 12.2.8. □

## 12.3 $\mathrm{Ex}^\infty(T)$ is always Kan

We begin the long road of proving $T \to \mathrm{Sing}\,|T|$ is always a weak equivalence. Our first stop is actually to understand $\mathrm{Ex}^\infty$ a little better, because Kan observed that $\mathrm{Ex}^\infty$ had some wonderful properties which will wind up helping us later. The first property is that $\mathrm{Ex}^\infty$ turns simplicial sets into Kan complexes; thus $\mathrm{Ex}^\infty$ actually gives us a rather explicit form of fibrant replacement.

**Ideation 12.3.1.** How do we prove that $\mathrm{Ex}^\infty(T)$ is Kan? We need to prove that every map $\Lambda^i_n \to \mathrm{Ex}^\infty(T)$ extends to $\Delta_n$.

As $\Lambda^i_n$ is finite, this map factors through a finite stage, so we can treat it as a map $\Lambda^i_n \to \mathrm{Ex}^m(T)$. The 'best possible case' would be if this map already extended to $\Delta_n$; but it seems unlikely that $\mathrm{Ex}^m(T)$ is already a Kan complex, so probably we need to hope that this map does not directly extend, but perhaps it extends if we replace $\mathrm{Ex}^m(T)$ by $\mathrm{Ex}^{m'}(T)$ for some $m' > m$.

The next 'best possible case' would be if we could get away with replacing $m$ by $m+1$.

Setting $T' = \mathrm{Ex}^m(T)$, this suggests we study the pushout diagram

$$\begin{array}{ccc} \mathrm{sd}(\Lambda^i_n) & \longrightarrow & \Lambda^i_n \\ \downarrow & & \downarrow \\ \mathrm{sd}(\Delta_n) & \longrightarrow & S, \end{array}$$

as extending our map $\Lambda^i_n \to \mathrm{Ex}(T')$ to $\Delta_n$ is equivalent to extending a map $\mathrm{sd}(\Lambda^i_n) \to T'$ to $\mathrm{sd}(\Delta_n)$, assuming our map $\mathrm{sd}(\Lambda^i_n) \to T'$ factors through $\Lambda^i_n$.

By Proposition 11.5.8, the map $\Lambda^i_n \to S$ is strongly anodyne. However, this is not quite enough to conclude: we don't know $T'$ is a Kan complex, so to get every map $\Lambda^i_n$ to extend to $S$ (and hence to $\mathrm{sd}(\Delta_n)$), we really want to say that $\Lambda^i_n \to S$ admits a retraction.

This is equivalent to giving a map

$$r : \mathrm{sd}(\Delta_n) \to \Lambda^i_n$$

which, restricted to $\mathrm{sd}(\Lambda^i_n)$, is the last vertex map. Unfortunately, such a map need not exist when $n > 1$. However, if we subdivide one more time, we can create such a map.

**Lemma 12.3.2.** *For every $i, n$, the map $\mathrm{sd}(\lambda_{\Lambda^i_n}) : \mathrm{sd}^2(\Lambda^i_n) \to \mathrm{sd}(\Lambda^i_n)$ extends to $\mathrm{sd}^2(\Delta_n)$.*

*Proof.* Set $C_n$ the poset of nonempty subsets of $[n]$. Set $C^i_n = C_n \setminus \{[n], [n] \setminus \{i\}\}$.

Then $\mathrm{sd}(\Lambda^i_n) = N(C^i_n)$, and $\mathrm{sd}^2(\Delta_n) = N(P_n)$ for $P_n$ the poset of chains in $C_n$. Similarly, $\mathrm{sd}^2(\Lambda^i_n) = N(P'_n)$, where $P'_n \subseteq P_n$ is the subset of chains not containing $[n]$ or $[n] \setminus \{i\}$.

The map

$$\mathrm{sd}(\lambda_{\Lambda^i_n}) : \mathrm{sd}^2(\Lambda^i_n) \to \mathrm{sd}(\Lambda^i_n)$$

is induced by the nerve of the monotone map

$$(S_0 \subseteq \cdots \subseteq S_k) \mapsto \{\max(S_0), \max(S_1), ..., \max(S_k)\}.$$

We wish to extend this to a monotone map on all chains. The trouble with keeping this formula for chains where the $S_i$ are allowed to be $[n]$ or $[n] \setminus \{i\}$ is that the resulting chain of maximums must lie in $C^i_n$, yet it's possible for this to fail in general; for example the chain

$$\{0\} \subseteq \{0, 1\} \subseteq \cdots \subseteq \{0, ..., n\}$$

is sent by this maximum map to $[n]$. To fix this, we just change the maximum of $\{0, ..., n\}$ to be something else by modifying the maximum function.

Similarly, the chain $[n] \setminus \{i\}$ can sometimes be produced by this procedure, for example if we plug in the chain

$$[0] \setminus \{i\} \subseteq [1] \setminus \{i\} \subseteq \cdots \subseteq [n] \setminus \{i\}.$$

To fix this, we modify the maximum function to be

$$m(S) := \begin{cases} \max(S) & S \notin \{[n], [n] \setminus \{i\}\}, \\ i & \text{otherwise.} \end{cases}$$

We chose this value $i$ so that $[n] \setminus \{i\}$ will always be sent to $i$, preventing us from ever outputting $[n] \setminus \{i\}$.

Define then

$$r : \mathrm{sd}^2(\Delta_n) \to \mathrm{sd}(\Lambda^i_n)$$

as the nerve of the monotone map

$$(S_0 \subseteq \cdots \subseteq S_k) \mapsto \{m(S_0), ..., m(S_k)\}.$$

The monotonicity is clear (as both sides are ordered by inclusion), as is agreement on $\mathrm{sd}^2(\Lambda^i_n)$.

To see that it is actually valued in $\mathrm{sd}(\Lambda^i_n)$, we must prove that $\{m(S_0), ..., m(S_k)\}$ is never $[n]$ or $[n] \setminus \{i\}$.

We will say $S_j$ is *exceptional* if $S_j \in \{[n], [n] \setminus \{i\}\}$. If all of the $S_j$ are exceptional, then $\{m(S_0), ..., m(S_k)\} = \{i\}$, which is not $[n]$ or $[n] \setminus \{i\}$, and so we're happy.

Otherwise, let $j$ be the largest index of a non-exceptional set. Then $S_j$ is neither $[n]$ nor $[n] \setminus \{i\}$, and so it omits some $i' \neq i$. Every non-exceptional member is contained in $S_j$, and so its maximum is not $i'$. But exceptional members always have $m = i$, and so it is impossible for $i'$ to appear in our $\{m(S_0), ..., m(S_k)\}$ chain, implying that our chain is not $[n]$ or $[n] \setminus \{i\}$, as desired. □

**Theorem 12.3.3** (Kan)**.** *Let $T$ be a simplicial set. Then $\mathrm{Ex}^\infty(T)$ is a Kan complex.*

*Proof.* Assume $\Lambda^i_n \to \mathrm{Ex}^\infty(T)$ is a horn. Then it factors through some $T' := \mathrm{Ex}^m(T)$. Choosing $m$ large enough, we can assume $m > 1$.

Define $S$ by the pushout diagram

$$\begin{array}{ccc} \mathrm{sd}(\Lambda^i_n) & \longrightarrow & \Lambda^i_n \\ \downarrow & & \downarrow \\ \mathrm{sd}(\Delta_n) & \longrightarrow & S. \end{array}$$

By Lemma 12.3.2, as sd commutes with colimits so that $\mathrm{sd}(S)$ is still a pushout, we deduce that $\mathrm{sd}(\Lambda^i_n) \to \mathrm{sd}(S)$ admits a retraction.

Our horn $\alpha : \Lambda^i_n \to \mathrm{Ex}^m(T)$ gives rise to a map $\alpha' : \mathrm{sd}(\Lambda^i_n) \to \mathrm{Ex}^{m-1}(T)$. As $\mathrm{sd}(\Lambda^i_n) \to \mathrm{sd}(S)$ admits a retraction, this $\alpha'$ extends to $\beta' : \mathrm{sd}(S) \to \mathrm{Ex}^{m-1}(T)$. Restricting, we find that $\beta'$ gives rise to a map $\beta'' : \mathrm{sd}^2(\Delta_n) \to \mathrm{Ex}^{m-1}(T)$, and hence to a map

$$\beta : \Delta_n \to \mathrm{Ex}^{m+1}(T)$$

which evidently extends $\rho_{\mathrm{Ex}^m(T)} \circ \alpha$. Thus $\mathrm{Ex}^\infty(T)$ is a Kan complex, as desired. □

## 12.4 Using $\mathrm{Ex}^\infty$ as fibrant replacement

By Theorem 12.3.3, for every simplicial set $T$, the map $T \to \mathrm{Ex}^\infty(T)$ is a weak equivalence and an inclusion whose target is a Kan complex. This is exactly the kind of thing that our earlier fibrant replacement theorem produced! Kan observed that $\mathrm{Ex}^\infty$ is actually an incredibly convenient fibrant replacement functor: it allows us to turn a Kan fibration into a *Kan fibration of Kan complexes* in a way that's compatible with finite limits – in particular, fibers! This allows us to prove general variants of all of our earlier results on Kan fibrations of Kan complexes.

To begin these applications, we start with the easy result making them all work.

**Proposition 12.4.1.** *Let $f : X \to S$ be a Kan fibration of simplicial sets. Then $\mathrm{Ex}^\infty(f)$ is a Kan fibration of Kan complexes, and the diagram*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Ex}^\infty(X) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Ex}^\infty(S) \end{array}$$

*commutes, with both horizontal maps being weak equivalences.*

*Proof.* The only new part is that $\mathrm{Ex}^\infty(f)$ is a Kan fibration. By Corollary 11.5.9, each $\mathrm{Ex}^n(f)$ is a Kan fibration; so, we need only a limiting argument. But for any horn-lifting square for $\mathrm{Ex}^\infty(f)$, finiteness lets us factor the horn and simplex through some common $\mathrm{Ex}^n(f)$, and then we fill the horn there. □

### 12.4.1 Trivial Kan fibrations

**Theorem 12.4.2.** *Let $f : X \to S$ be a Kan fibration. Then the following are equivalent:*

1. *$f$ is a homotopy equivalence,*
2. *$f$ is a weak equivalence,*
3. *$f$ is a trivial Kan fibration.*

*Proof.* Of course 1 implies 2. 3 implies 1 by Theorem 4.8.14. All that remains is to show 2 implies 3, so assume $f$ is a weak equivalence.

Then $\mathrm{Ex}^\infty(f)$ is a weak equivalence as well (by the 2-out-of-3 rule, as $X \to \mathrm{Ex}^\infty(X)$ and $S \to \mathrm{Ex}^\infty(S)$ are both weak equivalences); by Corollary 7.4.2, $\mathrm{Ex}^\infty(f)$ is therefore a trivial Kan fibration.

As $\mathrm{Ex}^\infty$ commutes with finite limits (Ex commutes with arbitrary limits and $\mathrm{Ex}^\infty$ is defined using a filtered colimit), we deduce that

$$\mathrm{Ex}^\infty(X_s) = \mathrm{Ex}^\infty(X) \times_{\mathrm{Ex}^\infty(S)} \rho_\infty(s).$$

In particular, as $\mathrm{Ex}^\infty(f)$ is a trivial Kan fibration, this $\mathrm{Ex}^\infty(X_s)$ is a contractible Kan complex. Thus $X_s$ is also a contractible Kan complex (as $X_s \to \mathrm{Ex}^\infty(X_s)$ is a weak equivalence of Kan complexes).

Assume we have a commutative diagram

$$\begin{array}{ccc} \partial\Delta_n & \xrightarrow{a} & X \\ \downarrow & & \downarrow f \\ \Delta_n & \xrightarrow{\bar{b}} & S. \end{array}$$

As $\mathrm{pt} \hookrightarrow \Delta_n$ is a homotopy equivalence (see Theorem 8.3.2), we may find some homotopy

$$\bar{H} : \Delta_n \times \Delta_1 \to S$$

from $\bar{b}$ to a constant map, say with value $s$. As $[\partial\Delta_n, X] \to [\partial\Delta_n, S]$ is a Kan fibration, this homotopy can be lifted to a compatible homotopy

$$H_0 : \partial\Delta_n \times \Delta_1 \to X$$

with left endpoint $a$.

Then $a_1 := H_0|_{\partial\Delta_n \times \{1\}}$ is a map $\partial\Delta_n \to X_s$, and so the contractibility of $X_s$ allows us to extend $a_1$ to a map $b_1 : \Delta_n \to X_s$. This isn't quite a lift of $\bar{b}$, though, so we need to go backwards along our homotopy.

As

$$(\partial\Delta_n \times \Delta_1) \coprod_{\partial\Delta_n \times \{1\}} (\Delta_n \times \{1\}) \hookrightarrow \Delta_n \times \Delta_1$$

is anodyne, we may find a map

$$H : \Delta_n \times \Delta_1 \to X$$

which is $H_0$ on $\partial\Delta_n \times \Delta_1$ and which is $b_1$ on $\Delta_n \times \{1\}$, and so that $f \circ H = \bar{H}$. It follows that $H|_{\Delta_n \times \{0\}}$ is a lift of $\bar{b}$. □

### 12.4.2 Structure of anodyne maps

In Proposition 8.4.1, we proved that if $i : A \hookrightarrow B$ was a weak equivalence and an inclusion, then we could solve all lifting problems of Kan fibrations *of Kan complexes* against $i$. Using $\mathrm{Ex}^\infty$, we can generalize this as follows.

**Proposition 12.4.3.** *Let $i : A \hookrightarrow B$ be monic and a weak equivalence. Then $i$ is anodyne; in fact, it is even strongly anodyne.*

*Proof.* By fibrant replacement, we can factor $i$ as

$$A \hookrightarrow A' \xrightarrow{f} B,$$

where $i' : A \hookrightarrow A'$ is strongly anodyne and $f$ is a Kan fibration. By the 2-out-of-3 rule, $f$ is a weak equivalence, and so a trivial Kan fibration by Theorem 12.4.2. Thus it admits a section $s : B \to A'$, and in fact by lifting against

$$\begin{array}{ccc} A & \xrightarrow{i'} & A' \\ \downarrow{\scriptstyle i} & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\mathrm{id}_B} & B, \end{array}$$

we can even choose this section $s$ such that $s \circ i = i'$. Thus the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{s} & A' & \xrightarrow{f} & B, \end{array}$$

witnesses that $i$ is a retract of $i'$, and hence that $i$ is strongly anodyne. □

**Corollary 12.4.4** (Structure of anodyne maps)**.** *Let $i : A \to B$ be a morphism of simplicial sets. Then the following are equivalent:*

1. *$i$ is strongly anodyne,*
2. *$i$ is anodyne,*
3. *$i$ is monic and a weak equivalence.*

## 12.5 Excision

Let $X$ be a point-set topological space, and suppose $X = U \cup V$ is a union of two open subsets. Can we describe the simplicial set $\mathrm{Sing}(X)$ in terms of $\mathrm{Sing}(U)$ and $\mathrm{Sing}(V)$? In this sense, no: we cannot distinguish $X$ and $U \sqcup V$ from the information of just $\mathrm{Sing}(U)$ and $\mathrm{Sing}(V)$, so really we should ask if we can recover $\mathrm{Sing}(X)$ from the diagram

$$\begin{array}{ccc} & \mathrm{Sing}(U \cap V) & \\ \swarrow & & \searrow \\ \mathrm{Sing}(U) & & \mathrm{Sing}(V). \end{array}$$

**Ideation 12.5.1.** The 'best hope' is that $\mathrm{Sing}(X)$ is the pushout of the above diagram. However, this does not quite work.

First, observe that there is a natural map

$$\mathrm{Sing}(U) \coprod_{\mathrm{Sing}(U \cap V)} \mathrm{Sing}(V) \to \mathrm{Sing}(X).$$

This map is an *inclusion*, but it is not always surjective on simplices. Recall that an $n$-simplex of $\mathrm{Sing}(X)$ is just a continuous map $\sigma : \Delta^n \to X$. And unfortunately, such a map $\sigma$ need not have its image contained in either $U$ or $V$.

However, as $\Delta^n$ is a compact metric space, the Lebesgue number lemma tells us that we can subdivide $\Delta^n$ enough times such that each sub-simplex is mapped to either $U$ or $V$. This inspires the following technical lemma, and its proof.

### 12.5.1 A comparison result

**Proposition 12.5.2.** *Let $X$ be a point-set topological space, and suppose $\mathcal{U} = \{A_i\}_{i \in I}$ is a family of subsets whose interiors $U_i$ cover $X$. Define $\mathrm{Sing}^{\mathcal{U}}(X)$ the simplicial subset of $\mathrm{Sing}(X)$ which on $n$-simplices consists of maps $\Delta^n \to X$ whose image is contained in one of the $A_i$.*

*Then the inclusion*

$$i : \mathrm{Sing}^{\mathcal{U}}(X) \hookrightarrow \mathrm{Sing}(X)$$

*is a weak equivalence.*

**Warning 12.5.3.** While $\mathrm{Sing}(X)$ is a Kan complex, $\mathrm{Sing}^{\mathcal{U}}(X)$ is not always a Kan complex.

**Ideation 12.5.4.** How might we prove Proposition 12.5.2?

Let's ignore the subtleties of weak equivalences for now and just imagine how we might try to construct a homotopy inverse

$$r : \mathrm{Sing}(X) \to \mathrm{Sing}^{\mathcal{U}}(X).$$

Thus we want to somehow take every simplex of $X$ and send it to one contained in some $U_i$.

One thing we might attempt is subdivision: by the Lebesgue number lemma, we can always subdivide our simplex repeatedly until each subdivision lies in some $U_i$. This suggests that it might be more convenient to work with $\mathrm{Ex}^\infty(\mathrm{Sing}(X))$ rather than $\mathrm{Sing}(X)$ directly.

So, let's instead try proving that

$$j : \mathrm{Ex}^\infty(\mathrm{Sing}^{\mathcal{U}}(X)) \hookrightarrow \mathrm{Ex}^\infty(\mathrm{Sing}(X))$$

is a homotopy equivalence of Kan complexes.

The nice thing is that our subdivision arguments above suggest that it's easy to map *in* to these spaces; and one can prove a map of *Kan complexes* is a homotopy equivalence by testing against maps in. In fact, by Whitehead's theorem, we need to only check this for finite braced models of pointed spheres.

The workhorse of our argument is the following lemma, which says that, at least after subdividing $K$ a lot, maps $K \to \mathrm{Sing}(X)$ factor through $\mathrm{Sing}^{\mathcal{U}}(X)$, at least up to homotopy.

**Lemma 12.5.5.** *Take $K$ a finite,* braced *simplicial set. Take*

$$\sigma : K \to \mathrm{Sing}(X)$$

*a map, adjunct to the continuous function $f_\sigma : |K| \to X$.*

*Suppose that $L \subseteq K$ is a simplicial subset, with $\sigma(L) \subseteq \mathrm{Sing}^{\mathcal{U}}(X)$. Let*

$$\tau : L \to \mathrm{Sing}^{\mathcal{U}}(X)$$

*denote the associated restriction.*

*By the Lebesgue number lemma and finiteness of $K$, we can find some $r$ such that every simplex of $|\mathrm{sd}^r K| \simeq |K|$ is mapped via $f_\sigma$ into some $A_i$. This gives us a map*

$$f_{\sigma,r} : |\mathrm{sd}^r K| \simeq |K| \xrightarrow{f_\sigma} X$$

*which induces a map of simplicial sets*

$$\sigma_r : \mathrm{sd}^r(K) \to \mathrm{Sing}^{\mathcal{U}}(X).$$

*Then the diagram*

$$\begin{array}{ccc} \mathrm{sd}^r(K) & \xrightarrow{\sigma_r} & \mathrm{Sing}^{\mathcal{U}}(X) \\ \downarrow{\scriptstyle \lambda_K^{(r)}} & & \downarrow{\scriptstyle i} \\ K & \xrightarrow{\sigma} & \mathrm{Sing}(X). \end{array}$$

*is homotopy commutative. Moreover, we can choose the homotopy*

$$H : \mathrm{sd}^r(K) \times \Delta_1 \to \mathrm{Sing}(X)$$

*such that, on $\mathrm{sd}^r(L) \times \Delta_1$, it factors through $\mathrm{Sing}^{\mathcal{U}}(X)$.*

*Proof.* Let

$$\beta_r : |\mathrm{sd}^r(K)| \simeq |K|$$

be the barycentric subdivision homeomorphism associated to this braced simplicial set.

Then $\sigma \circ \lambda_K^{(r)}$ is adjunct to the continuous map $f_\sigma \circ |\lambda_K^{(r)}|$, and $i \circ \sigma_r$ is adjunct to the continuous map $f_\sigma \circ \beta_r$.

By Theorem 3.6.9, a continuous homotopy between $f_\sigma \circ \beta_r$ and $f_\sigma \circ |\lambda_K^{(r)}|$ will give a simplicial homotopy from $\sigma \circ \lambda_K^{(r)}$ to $i \circ \sigma_r$.

To produce such a continuous homotopy, it suffices to check that $\beta_r$ and $|\lambda_K^{(r)}|$ are homotopic. Fortunately, this is possible via a straight-line homotopy

$$J : |\operatorname{sd}^r(K)| \times [0,1] \to |K|,$$

constructed as follows: for each non-degenerate $n$-simplex $\gamma$ of $K$, as $\gamma$ is convex, the maps $\beta_r(\gamma)$ and $\lambda_\gamma^{(r)}$ are homotopic by a straight-line homotopy; we can then glue these straight-line homotopies together to get our $J$.

Put $\tilde{H} := f_\sigma \circ J$. To conclude, we must understand the restriction of $\tilde{H}$ to $|\operatorname{sd}^r(L)| \times [0,1]$.

For any simplex $\gamma$ of $L$, and a point $x \in \gamma$, the homotopy $\tilde{H}$, at $x$, gives a path from $f_\sigma(\beta_r(x))$ to $f_\sigma(|\lambda_K^{(r)}|(x))$ contained entirely inside of $f_\sigma(\gamma)$. By our assumption on $L$, we know $f_\sigma(\gamma) \subseteq A_i$ for some $i$. Thus the adjunct $H$ to $\tilde{H}$, restricted to $\operatorname{sd}^r(L) \times \Delta_1$, is valued in $\operatorname{Sing}^{\mathcal{U}}(X)$, as desired. $\square$

*Proof of Proposition 12.5.2.* Let $j := \operatorname{Ex}^\infty(i)$. It suffices to prove $j$ is a weak equivalence by the 2-out-of-3 rule, since $S \to \operatorname{Ex}^\infty(S)$ is always a weak equivalence.

As $\operatorname{sd}^\infty(\mathrm{pt}) = \mathrm{pt}$, we find

$$\operatorname{Ex}^\infty(\operatorname{Sing}^{\mathcal{U}}(X))_0 = \operatorname{Ex}^\infty(\operatorname{Sing}(X))_0 = X,$$

the set of points of $X$. Choose $x \in X$ your favorite basepoint, and regard $\operatorname{Ex}^\infty(\operatorname{Sing}(X))$ and $\operatorname{Ex}^\infty(\operatorname{Sing}^{\mathcal{U}}(X))$ as pointed simplicial sets.

To begin, we prove that if $(K,k)$ is a pointed finite braced simplicial set, then

$$\pi_0[(K,k), \operatorname{Ex}^\infty(\operatorname{Sing}^{\mathcal{U}}(X))]_* \xrightarrow{j_*} \pi_0[(K,k), \operatorname{Ex}^\infty(\operatorname{Sing}(X))]_* \tag{12.5.6}$$

is a bijection.

For surjectivity, assume we are given some pointed map

$$\sigma : K \to \operatorname{Ex}^\infty(\operatorname{Sing}(X)).$$

As $K$ is finite, this factors through some $\operatorname{Ex}^n(\operatorname{Sing}(X))$, and hence comes (after adjunction) from some map

$$\sigma' : \operatorname{sd}^n(K) \to \operatorname{Sing}(X).$$

This $\operatorname{sd}^n(K)$ is still finite and braced. Applying Lemma 12.5.5 (with $L = \emptyset$) to $\sigma'$, we find that up to homotopy, it factors through $\operatorname{Sing}^{\mathcal{U}}(X)$. Thus there is a map

$$\tau' : \operatorname{sd}^{n+r}(K) \to \operatorname{Sing}^{\mathcal{U}}(X)$$

with $\sigma' \circ \lambda^{(r)}_{\operatorname{sd}^n(K)} \simeq i \circ \tau'$. In Lemma 12.5.5, the map $\tau'$ is produced in such a way that it agrees with $\sigma' \circ \lambda^{(r)}_{\operatorname{sd}^n(K)}$ on $k$, and so $\tau'$ is still pointed; moreover, as our homotopy was a straight-line homotopy obtained by glueing things on simplices, it is in fact a pointed homotopy of pointed maps. As the $(\operatorname{sd}, \operatorname{Ex})$ adjunction preserves homotopic maps, we deduce that (12.5.6) is surjective.

For injectivity, we again apply Lemma 12.5.5, but this time with a clever choice of $L$.

Take

$$\sigma_0, \sigma_1 : K \to \operatorname{Ex}^\infty(\operatorname{Sing}^{\mathcal{U}}(X))$$

two pointed maps such that $j\sigma_0, j\sigma_1$ are pointed homotopic. Choose

$$H : \Delta_1 \times K \to \operatorname{Ex}^\infty(\operatorname{Sing}(X))$$

a homotopy. As $\Delta_1 \times K$ is still finite braced, using Lemma 12.5.5 as before (but this time we'll ignore the minor details about $\mathrm{sd}^r$ or $\mathrm{Ex}^\infty$, leaving them to the reader) we may choose a map

$$H' : \Delta_1 \times K \to \mathrm{Ex}^\infty(\mathrm{Sing}^{\mathcal{U}}(X))$$

such that $jH' \simeq H$, by a homotopy

$$\tilde{H} : \Delta_1 \times (\Delta_1 \times K) \to \mathrm{Ex}^\infty(\mathrm{Sing}(X))$$

which on $\{0,1\} \times K$ takes values in $\mathrm{Ex}^\infty(\mathrm{Sing}^{\mathcal{U}}(X))$ (here we're using $L = (\{0,1\} \times K) \cup (\Delta_1 \times \{k\})$ in Lemma 12.5.5).

Thus $H'$ is a pointed homotopy between its endpoints, which are themselves pointed homotopic to $\sigma_0, \sigma_1$, showing by transitivity that $\sigma_0 \simeq \sigma_1$, giving our desired injectivity.

Theorem 7.6.6 tells us homotopy groups can be computed using $\Delta_n/\partial\Delta_n$. For $n \geq 2$, this simplicial set is not braced; however, it is weakly equivalent to the braced simplicial set $\partial\Delta_{n+1}$ by the map

$$\partial\Delta_{n+1} \to \partial\Delta_{n+1}/\Lambda^{n+1}_{n+1} \simeq \Delta_n/\partial\Delta_n,$$

using Theorem 8.2.4. Thus the work above, combined with Whitehead's theorem, implies $j$ is a homotopy equivalence of Kan's complex. □

### 12.5.2 Excision

The following special case of Proposition 12.5.2 is very useful in practice.

**Theorem 12.5.7** (Excision)**.** *Let $X$ be a point-set topological space, and suppose*

$$X = A \cup B$$

*where $A, B$ are subsets whose interiors also cover $X$. Then the natural map*

$$\mathrm{Sing}(A) \sqcup_{\mathrm{Sing}(A \cap B)} \mathrm{Sing}(B) \to \mathrm{Sing}(X)$$

*is a weak equivalence of simplicial sets.*

*Proof.* Apply Proposition 12.5.2 to $\{A, B\}$. □

# Chapter 13

# Milnor's theorem and simplicial approximation

## 13.1 Introduction

We finally have enough machinery to prove Milnor's theorem, which has been oft alluded to throughout this text.

**Theorem 13.1.1** (Milnor's theorem)**.** *Let $X$ be a simplicial set. Then the unit map $u_X : X \to \operatorname{Sing}|X|$ is a weak equivalence.*

## 13.2 The proof of Milnor's theorem

Fix $X$ a simplicial set. We will prove that

$$u_X : X \to \operatorname{Sing}|X|$$

is a weak equivalence.

It's easy to check Theorem 13.1.1 holds for standard simplices.

**Lemma 13.2.1.** *If $X = \Delta_n$ is an $n$-simplex, then $u_X$ is a weak equivalence.*

*Proof.* By the 2-out-of-3 rule, since $\Delta_n \to \mathrm{pt}$ is a weak equivalence, it suffices to prove $\operatorname{Sing}\Delta^n$ is homotopic to the point. But $\Delta^n$ is homotopic to the point, so by Theorem 4.5.16 we deduce $\operatorname{Sing}\Delta^n$ is also contractible and hence conclude. □

We will prove $u_X$ is a weak equivalence by a devissage. This is aided by the fact that the unit of any adjunction is stable under colimits, in the following sense.

**Lemma 13.2.2.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint functor, with right adjoint $G : \mathcal{D} \to \mathcal{C}$. Suppose $X_i$ is an $I$-shaped diagram in $\mathcal{C}$, such that this diagram has a colimit $X$. Let $\eta_i : X_i \to GFX_i$ denote the unit map of the adjunction; then the maps*

$$\eta_i' : X_i \xrightarrow{\eta_i} GFX_i \to GF(\operatorname{colim}_i X_i)$$

*induce the unit map*

$$\eta : X \to GF(X).$$

*Proof.* For $c_i : X_i \to X$ the natural maps, we have

$$\eta_X \circ c_i = GF(c_i) \circ \eta_{X_i}$$

by naturality of $\eta$. □

### 13.2.1 Finite simplicial sets

To begin, assume $X$ is a *finite* simplicial set. By an easy induction argument, we will be able to prove $u_X$ is a weak equivalence in this case.

We proceed by induction on dimension. For the base case of $\dim X = -1$, we have $X = \emptyset$, in which case the theorem is true as $\mathrm{Sing}(|\emptyset|) = \emptyset$.

For the inductive step, take $X$ a non-empty finite simplicial set of dimension $d \geq 0$, and suppose we know Theorem 13.1.1 holds for every finite simplicial set of smaller dimension. We will further induct on the *number of non-degenerate simplices of dimension d*. The base case of zero such simplices follows from the induction hypothesis for simplicial sets of dimension $d - 1$.

**Lemma 13.2.3.** *Let $X$ be a finite simplicial set of dimension $d \geq 0$. Then we can find some simplicial subset $X'$ of $X$, and some non-degenerate $d$-simplex $\sigma \in X_d \setminus X'_d$, such that $X$ is given as the pushout*

$$\begin{array}{ccc} \partial\Delta_d & \longrightarrow & \Delta_d \\ \downarrow & & \downarrow{\scriptstyle \sigma} \\ X' & \longrightarrow & X. \end{array}$$

*Proof.* As $X$ is dimension $d$, we can find some non-degenerate $d$-simplex $\sigma$ of $X$. Define $X'$ the simplicial subset spanned by the simplices $X_0 \cup \cdots \cup X_{d-1} \cup (X_d \setminus \{\sigma\})$. In particular, as $\sigma$ is non-degenerate, $\sigma \notin X'_d$. It's immediate then that the diagram in the lemma's statement is a pushout. □

By Lemma 13.2.3, we can write $X = \Delta_d \coprod_{\partial\Delta_d} X'$, where Theorem 13.1.1 holds for $X'$ by the induction hypothesis and for $\Delta_d$ by Lemma 13.2.1. We also know that Theorem 13.1.1 holds for $\partial\Delta_d$, as it has dimension $d - 1$.

How can we leverage this situation? By Lemma 13.2.2, the map $u_X$ is induced by the morphism of spans

$$\begin{array}{ccccc} \Delta_d & \longleftarrow & \partial\Delta_d & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sing}(|\Delta_d|) & \longleftarrow & \mathrm{Sing}(|\partial\Delta_d|) & \longrightarrow & \mathrm{Sing}(|X'|), \end{array}$$

where we mean that this morphism of spans gives a map of pushouts

$$u'_X : X \to \mathrm{Sing}(|\Delta_d|) \coprod_{\mathrm{Sing}(|\partial\Delta_d|)} \mathrm{Sing}(|X'|),$$

and the natural map $\alpha$ from that pushout to $\mathrm{Sing}(|X|)$ satisfies $u_X = \alpha \circ u'_X$. By the 2-out-of-3 rule, then, it suffices to check that

1. $u'_X$ is a weak equivalence,

2. $\alpha : \mathrm{Sing}(|\Delta_d|) \coprod_{\mathrm{Sing}\,|\partial\Delta_d|} \mathrm{Sing}\,|X'| \to \mathrm{Sing}\,|X|$ is a weak equivalence.

At this point, the problem seems to be screaming 'homotopy pushouts' at us. Indeed, with our homotopy pushout machinery, it is immediate that $u'_X$ is a weak equivalence.

By Proposition 10.3.12, the map of homotopy pushouts

$$X' \overset{\mathrm{h}}{\coprod_{\partial\Delta_d}} \Delta_d \to \mathrm{Sing}(|X'|) \overset{\mathrm{h}}{\coprod_{\mathrm{Sing}(|\partial\Delta_d|)}} \mathrm{Sing}(|\Delta_d|)$$

is a weak equivalence. The diagram

$$\begin{array}{ccc} X' \coprod^{\mathrm{h}}_{\partial\Delta_d} \Delta_d & \longrightarrow & \mathrm{Sing}(|X'|) \coprod^{\mathrm{h}}_{\mathrm{Sing}(|\partial\Delta_d|)} \mathrm{Sing}(|\Delta_d|) \\ \downarrow & & \downarrow \\ X' \coprod_{\partial\Delta_d} \Delta_d & \xrightarrow{u'_X} & \mathrm{Sing}(|X'|) \coprod_{\mathrm{Sing}(|\partial\Delta_d|)} \mathrm{Sing}(|\Delta_d|) \end{array}$$

commutes, and Proposition 10.3.7 tells us that the two vertical arrows are weak equivalences. By the 2-out-of-3 rule, it follows that $u'_X$ is a weak equivalence, as desired.

It is a bit more annoying to prove that $\alpha$ is a weak equivalence, but we can do it.

**Ideation 13.2.4.** This $\alpha$ might remind us of excision; unfortunately, excision alone does not work because the interiors of $\Delta^d$ and $|X'|$ do not cover $|X|$, and because $\Delta^d$ is not a subset of $|X|$; it just continuously maps to $|X|$. The $|X'|$ situation is easier to fix. $|X'|$ was originally obtained by deleting the interior of some simplex; by a radial deformation retraction, $|X'|$ is homotopy equivalent to $V = |X| \setminus \{x\}$, where $x \in |X|$ is any point in the interior of the removed simplex.

The failure of

$$\Delta^d \to |X|$$

to be injective only occurs on the boundary, so we replace $\Delta^d$ with its interior $U$, which is an open subspace of $|X|$. As we expanded $|X'|$ to $V = |X| \setminus \{x\}$, the open sets $U, V$ now cover $|X|$.

Excision tells us then that the natural map

$$\beta : \mathrm{Sing}(U) \coprod_{\mathrm{Sing}(U \setminus \{x\})} \mathrm{Sing}(V) \to \mathrm{Sing}(|X|)$$

is a weak equivalence.

This is close to what we want, as $\alpha$ is a map

$$\mathrm{Sing}(\Delta^d) \coprod_{\mathrm{Sing}(\partial\Delta^d)} \mathrm{Sing}(|X'|) \to \mathrm{Sing}(|X|).$$

It is somewhat hard to compare $\alpha$ and $\beta$, as $U$ is smaller than $\Delta^d$ but $V$ is bigger than $|X'|$, so the two factors of our coproduct are going in different ways. However, we can create the space

$$S = \mathrm{Sing}(\Delta^d) \coprod_{\mathrm{Sing}(\Delta^d \setminus \{x\})} \mathrm{Sing}(V),$$

and the domains of $\alpha, \beta$ both naturally map to $S$.

As $\Delta^d$ is homotopy equivalent to $\mathrm{Sing}(U)$, and as $\mathrm{Sing}(V)$ and $\mathrm{Sing}(|X'|)$ are homotopy equivalent, it follows from Proposition 10.3.12 (together with the cylinder-collapse maps of Proposition 10.3.7) that the maps from the domains of $\alpha, \beta$ to $S$ are weak equivalences. Moreover, both $\alpha, \beta$ factor through the obvious map $S \to \mathrm{Sing}(X)$. As $\beta$ is a weak equivalence, using the 2-out-of-3 rule twice gives that $\alpha$ is a weak equivalence.

### 13.2.2 Finishing

**Lemma 13.2.5.** *Let $X$ be a simplicial set. Let $\mathcal{L}$ denote the lattice of* finite *simplicial subsets of $X$, ordered by inclusion. Then*

$$X = \mathrm{colim}_{A \in \mathcal{L}} A,$$

*so that every simplicial set is a filtered colimit of finite simplicial sets.*

*Proof.* Immediate from the pointwise formula for colimits and the corresponding fact about sets. □

*Proof of Theorem 13.1.1.* Write

$$X = \mathrm{colim}_{A \in \mathcal{L}} A$$

as a filtered colimit of finite simplicial sets, as in Lemma 13.2.5. As $\Delta^n$ is compact, any continuous map $\Delta^n \to |X|$ factors through some $|A|$, and hence

$$\mathrm{colim}_{A \in \mathcal{L}} \mathrm{Sing}(|A|) \to \mathrm{Sing}(|X|)$$

is an isomorphism of simplicial sets. By Lemma 13.2.2, the map $u_X$ is the filtered colimits of the maps $u_A$. As each $u_A$ is a weak equivalence (since we have already proven Milnor's theorem for finite simplicial sets), Theorem 8.5.4 implies that $u_X$ is a weak equivalence, as desired. □

## 13.3 Some applications

To end, we give some fun applications of Milnor's theorem.

### 13.3.1 Geometric realization detects weak equivalence

**Theorem 13.3.1.** *Let $f : S \to T$ be a morphism of simplicial sets. Then $f$ is a weak equivalence if and only if $|f|$ is a homotopy equivalence.*

*Proof.* The forward direction was proven in Proposition 8.6.1. For the converse, use 2-out-of-3 and the commutative square

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow{\scriptstyle u_S} & & \downarrow{\scriptstyle u_T} \\ \mathrm{Sing}(|S|) & \xrightarrow{\mathrm{Sing}(|f|)} & \mathrm{Sing}(|T|) \end{array}$$

to see that $f$ is a weak equivalence, as Theorem 4.5.16 says that when $|f|$ is a homotopy equivalence, then so is $\mathrm{Sing}\,|f|$. □

### 13.3.2 Simplicial approximation

We can finally prove a general simplicial approximation theorem.

**Theorem 13.3.2** (Simplicial approximation)**.** *Let $S$ be a finite simplicial set and $T$ a simplicial set. Given a continuous function*

$$f : |S| \to |T|,$$

*there exists some $N \gg 0$ and a simplicial set map $g : \mathrm{sd}^N(S) \to T$ such that*

$$|g| \simeq f \circ |\lambda_S^{(N)}|.$$

*Here, $\lambda_S^{(N)}$ is the composition $\lambda_S \circ \lambda_{\mathrm{sd}(S)} \circ \cdots \circ \lambda_{\mathrm{sd}^{N-1}(S)}$ of last vertex maps.*
*If $T$ is a Kan complex, we can even take $N = 0$.*

**Ideation 13.3.3.** The continuous function $f$, by adjunction, gives us a morphism of simplicial sets

$$h : S \to \mathrm{Sing}(|T|).$$

By Theorem 13.1.1, the unit map $u_T : T \to \mathrm{Sing}(|T|)$ is a weak equivalence. If $T$ were a Kan complex, then at this point we're done: there would be a (unique up to homotopy) map $h' : S \to T$ with $u_T \circ h' \simeq h$, and this map $h'$ would save the day. Unfortunately, $T$ is not always a Kan complex.

However, $\mathrm{Ex}^\infty(T)$ is Kan. Thus we may find a map

$$h' : S \to \mathrm{Ex}^\infty(T)$$

such that

$$\mathrm{Ex}^\infty(u_T) \circ h' \simeq i_{\mathrm{Sing}(|T|)} \circ h,$$

for $i_{\mathrm{Sing}(|T|)} : \mathrm{Sing}(|T|) \to \mathrm{Ex}^\infty(\mathrm{Sing}(|T|))$ the natural map.

This basically solves our problem, with a little bookkeeping.

*Proof.* Following Ideation 13.3.3, we produce $h' : S \to \mathrm{Ex}^\infty(T)$. As $S$ is finite, this $h'$ and the homotopy $\mathrm{Ex}^\infty(u_T) \circ h' \simeq i_{\mathrm{Sing}(|T|)} \circ h$ both factor through some $\mathrm{Ex}^N$, and so we can actually create some

$$h'_N : \mathrm{sd}^N(S) \to T$$

with

$$u_T \circ h'_N \simeq h \circ \lambda_S^{(N)}.$$

Take $g := h'_N$. By Corollary 4.5.20, we deduce that

$$|g| \simeq f \circ |\lambda_S^{(N)}|,$$

as desired. □

# Appendix A

# Producing anodyne morphisms

## A.1 Introduction

The goal of this appendix is to prove Theorem 4.8.8. Let's recall the setup.

Let $i : A \hookrightarrow B$ be an inclusion of simplicial sets, and let $j : C \hookrightarrow D$ be a *strongly* anodyne inclusion.

Theorem 4.8.8 states that

$$(A \times D) \coprod_{A \times C} (B \times C) \hookrightarrow B \times D \tag{A.1.1}$$

is also strongly anodyne.

## A.2 Digression: opposites

Before continuing, we make a brief digression on an operation not treated elsewhere in the text, but which somewhat simplifies our work below. This operation only plays a minor role in our argument, but because of the importance of this technique (despite us not needing it anywhere else in the text), we thought it worth mentioning.

**Definition A.2.1.** Let $S$ be a simplicial set. The *opposite simplicial set* $S^{\mathrm{op}}$ is the functor

$$S^{\mathrm{op}} : \Delta^{\mathrm{op}} \to \mathrm{Set}$$

obtained as the composition

$$S \circ \mathrm{op},$$

where $\mathrm{op} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is the functor sending a finite totally ordered set to its opposite.

**Proposition A.2.2.** *The functor $S \mapsto S^{\mathrm{op}}$ is an equivalence of categories.*

*Proof.* This functor is its own inverse! □

**Proposition A.2.3.** *If $\mathcal{C}$ is a category, then $N(\mathcal{C}^{\mathrm{op}}) = N(\mathcal{C})^{\mathrm{op}}$.*

*Proof.* Left to the reader as an exercise. □

**Proposition A.2.4.** *The opposite of $\Lambda^i_n \hookrightarrow \Delta_n$ is isomorphic (as a morphism) to $\Lambda^{n-i}_n \hookrightarrow \Delta_n$.*

*Proof.* The order reversing map

$$[n] \to [n],$$
$$i \mapsto n - i$$

gives a functor

$$[n]^{\mathrm{op}} \to [n],$$

and taking nerves this gives us an isomorphism

$$\Delta^{\mathrm{op}}_n \simeq \Delta_n.$$

It is easy to check that this isomorphism intertwines the two claimed subsets of $\Delta_n$. □

**Proposition A.2.5.** *If $f : X \to S$ is a Kan fibration, then $f^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a Kan fibration.*

*Proof.* Immediate from Proposition A.2.4. □

## A.3 Leibniz product

A morphism of the form appearing in (A.1.1) has a special name.

**Definition A.3.1.** The *Leibniz product* of two morphisms $f : X \to Y$ and $g : A \to B$ is the natural morphism

$$f \boxtimes g : (X \times B) \coprod_{X \times A} (Y \times A) \to Y \times B$$

**Remark A.3.2.** There is a canonical equivalence $f \boxtimes g \simeq g \boxtimes f$.

**Construction A.3.3** (Functoriality)**.** Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y', \end{array}$$

and fix a morphism $g : A \to B$.

From this diagram, we construct a commutative diagram

$$\begin{array}{ccc} (X \times B) \coprod_{X \times A} (Y \times A) & \xrightarrow{\tilde{i}} & (X' \times B) \coprod_{X' \times A} (Y' \times A) \\ \downarrow{\scriptstyle f \boxtimes g} & & \downarrow{\scriptstyle f' \boxtimes g} \\ Y \times B & \xrightarrow{j \times \mathrm{id}_B} & Y' \times B \end{array}$$

as follows. Let

$$\tilde{i}(x, b) = (i(x), b),$$
$$\tilde{i}(y, a) = (j(y), a).$$

These agree on $X \times A$, because

$$\tilde{i}(x, g(a)) = (i(x), g(a))$$

and

$$\tilde{i}(f(x), a) = (j(f(x)), a) = (f'(i(x)), a),$$

and in $(X' \times B) \coprod_{X' \times A} (Y' \times A)$, we have

$$(f'(i(x)), a) = (i(x), g(a))$$

because both come from $(i(x), a)$.

It's easy to see the resulting diagram commutes. Moreover, a similar functoriality holds in the second argument, as can be obtained using the symmetry $f \boxtimes g \simeq g \boxtimes f$.

The Leibniz product is called a product because it is associative.

**Proposition A.3.4.** *Let $\{f_i : X_i \hookrightarrow Y_i\}_{i=1,2,3}$ be three monomorphisms of simplicial sets. Then*

$$(f_1 \boxtimes f_2) \boxtimes f_3$$

*and*

$$f_1 \boxtimes (f_2 \boxtimes f_3)$$

*are both the inclusion of*

$$(X_1 \times Y_2 \times Y_3) \cup (Y_1 \times X_2 \times Y_3) \cup (Y_1 \times Y_2 \times X_3)$$

*into $Y_1 \times Y_2 \times Y_3$.*

**Remark A.3.5.** We think of the source of $f_1 \boxtimes f_2 \boxtimes f_3$ as being the set of all triples $(y_1, y_2, y_3) \in Y_1 \times Y_2 \times Y_3$ where $y_i \in X_i$ for at least one index $i$.

*Proof.* In general, the Leibniz product of two inclusions $i : A \hookrightarrow B, i' : A' \hookrightarrow B'$ is just the inclusion

$$(A \times B') \cup (B \times A') \hookrightarrow B \times B',$$

from which this follows immediately. □

We can now rephrase our goal as follows: for every inclusion $i$ and strongly anodyne inclusion $j$, we want to prove that $i \boxtimes j$ is strongly anodyne.

We will prove this in two steps. First, we show that, for a fixed $i$, the class of inclusions $j$ for which $i \boxtimes j$ is strongly anodyne has very good stability properties. Then, we show that, when $j$ is a horn inclusion, $i \boxtimes j$ is strongly anodyne.

## A.4 Stability properties of Leibniz products

The goal of this section is to prove the following lemma.

**Lemma A.4.1.** *Let $i : A \hookrightarrow B$ be an inclusion of simplicial sets. Then the class of all inclusions $j : C \hookrightarrow D$ such that $i \boxtimes j$ is strongly anodyne is stable under retracts, pushouts, compositions, $\omega$-compositions, and arbitrary coproducts.*

There are several parts to this claim; we treat them in order. The first part is simple: stability under retracts follows from functoriality of $j \mapsto i \boxtimes j$ (recall Construction A.3.3).

For pushouts, we use the following lemma.

**Lemma A.4.2** (Pushouts)**.** *Let $i : A \hookrightarrow A'$ be an inclusion of simplicial sets. Suppose*

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B_2 \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ B_1' & \xrightarrow{g} & B_2' \end{array}$$

*is a pushout diagram, where $j_1$ is an inclusion (and hence $j_2$ is as well).*

*Then*

$$\begin{array}{ccc} s(i \boxtimes j_1) & \xrightarrow{\varphi} & s(i \boxtimes j_2) \\ \downarrow{\scriptstyle i \boxtimes j_1} & & \downarrow{\scriptstyle i \boxtimes j_2} \\ A' \times B_1' & \xrightarrow{\mathrm{id}_{A'} \times g} & A' \times B_2' \end{array}$$

*is a pushout diagram as well, where $s$ denotes the source of a morphism, and*

$$\varphi : (A \times B_1') \cup (A' \times B_1) \to (A \times B_2') \cup (A' \times B_2)$$

*is the map*

$$\varphi(a, b_1') = (a, g(b_1')), \varphi(a', b_1) = (a', f(b_1)).$$

*Proof.* As colimits in sSet are computed pointwise, it suffices to check this in the category of sets. We check that $A' \times B_2'$ obeys the correct universal property. Write $B_2' = B_1' \coprod_{B_1} B_2$.

Suppose we have maps

$$p : s(i \boxtimes j_2) \to T,$$

$$q : A' \times B_1' \to T,$$

which agree on $(A \times B_1') \cup (A' \times B_1)$.

Define

$$c : A' \times B_2' \to T$$

by

$$c(a', b_1') = q(a', b_1'), c(a', b_2) = p(a', b_2).$$

These two formulas agree for inputs $(a', b_1)$, thanks to the compatibility condition on $p$ and $q$. Hence $c$ is well-defined.

We wish to prove that this assignment $(p, q) \mapsto c$ is a bijection. Injectivity is clear. For surjectivity, let $c : A' \times B_2' \to T$ be given. Define

$$p : s(i \boxtimes j_2) = (A \times B_2') \cup (A' \times B_2) \to T$$

by

$$p := c|_{s(i \boxtimes j_2)},$$

and define

$$q : A' \times B_1' \to T$$

by

$$q := c \circ (\mathrm{id}_{A'} \times g).$$

We conclude. □

Composition is a little more subtle.

**Ideation A.4.3.** Let $i : A \hookrightarrow A'$ be a fixed inclusion, and consider two maps

$$j : B \hookrightarrow B', j' : B' \hookrightarrow B''.$$

How does $i \boxtimes (j' \circ j)$ relate to $i \boxtimes j$ and $i \boxtimes j'$?
We seek to relate the inclusion

$$(j' \circ j) \boxtimes i : (B \times A') \cup (B'' \times A) \hookrightarrow B'' \times A'$$

to the inclusions

$$j \boxtimes i : (B \times A') \cup (B' \times A) \hookrightarrow B' \times A'$$

and

$$j' \boxtimes i : (B' \times A') \cup (B'' \times A) \hookrightarrow B'' \times A'.$$

The map $j' \boxtimes i$ has the same target as $(j' \circ j) \boxtimes i$, but the wrong source; its source is slightly too large. However, we *can* write a pushout diagram

$$\begin{array}{ccc} (B \times A') \cup (B' \times A) & \longrightarrow & (B \times A') \cup (B'' \times A) \\ \downarrow{\scriptstyle j \boxtimes i} & & \downarrow \\ B' \times A' & \longrightarrow & (B' \times A') \cup (B'' \times A), \end{array}$$

so that $(j' \circ j) \boxtimes i$ is the *composition* of $j' \boxtimes i$ and a pushout of $j \boxtimes i$.
In particular, if $j \boxtimes i, j' \boxtimes i$ are both strongly anodyne, then $(j' \circ j) \boxtimes i$ is strongly anodyne as well.

From here, coproducts and $\omega$-compositions are easy to treat, so we can prove the entirety of Lemma A.4.1.

*Proof of Lemma A.4.1.* The resulting class is closed under retracts by functoriality of Leibniz products, closed under pushouts by Lemma A.4.2, and closed under composition by Ideation A.4.3.

As Leibniz products are compatible with arbitrary coproducts (they are built out of products and colimits, and in sSet products commute with arbitrary coproducts), we also find that this class is stable under arbitrary coproducts.

Finally, we check stability under $\omega$-compositions. Let

$$C_0 \hookrightarrow C_1 \hookrightarrow C_2 \hookrightarrow \cdots$$

be a chain of inclusions $j_n : C_n \hookrightarrow C_{n+1}$, such that each $i \boxtimes j_n$ is strongly anodyne.

Let $j : C_0 \hookrightarrow C_\omega$ be the $\omega$-composition. To see that

$$i \boxtimes j : (C_0 \times B) \coprod_{C_0 \times A} (C_\omega \times A) \hookrightarrow C_\omega \times B$$

is strongly anodyne, we use a trick similar to Ideation A.4.3. Put

$$E_n := (C_n \times B) \coprod_{C_n \times A} (C_\omega \times A).$$

Then

$$\operatorname{colim}_n E_n = C_\omega \times B,$$

but the $E_n$ can be described as pushouts

$$\begin{array}{ccc} (C_n \times B) \coprod_{C_n \times A} (C_{n+1} \times A) & \longrightarrow & E_n \\ \downarrow{\scriptstyle i \boxtimes j_n} & & \downarrow \\ C_{n+1} \times B & \longrightarrow & E_{n+1}, \end{array}$$

and in particular each $E_n \to E_{n+1}$ is strongly anodyne. As the class of strongly anodyne morphisms is stable under $\omega$-compositions, we conclude. □

## A.5 Prisms

As explained at the end of the last section, we start our proof of Theorem 4.8.8 in the case where $j$ is a horn inclusion. We begin with the simple horn inclusion $j : \{1\} \hookrightarrow \Delta_1$.

Let $i : A \hookrightarrow A'$ be an inclusion of simplicial sets. Then our Leibniz product is

$$i \boxtimes j : (A' \times \{1\}) \coprod_{A \times \{1\}} (A \times \Delta_1) \hookrightarrow A' \times \Delta_1.$$

**Ideation A.5.1.** Combining Lemma A.4.1 with the skeletal filtration (allowing us to build any simplicial set out of pushouts, coproducts, and $\omega$-compositions of inclusions $\partial\Delta_n \hookrightarrow \Delta_n$), to prove that $i \boxtimes j$ is strongly anodyne for every $i$, it is enough to prove $i \boxtimes j$ is strongly anodyne for each $i_n : \partial\Delta_n \hookrightarrow \Delta_n$.

### A.5.1 The simplest boundary inclusion

To whet our appetite, we explicitly prove $i_1 \boxtimes j$ is anodyne. Explicitly, this Leibniz product is the inclusion

$$i_1 \boxtimes j : (\{1\} \times \Delta_1) \coprod_{\{1\}\times\{0,1\}} (\Delta_1 \times \{0,1\}) \hookrightarrow \Delta_1 \times \Delta_1.$$

**Ideation A.5.2.** Set

$$A = (\{1\} \times \Delta_1) \coprod_{\{1\}\times\{0,1\}} (\Delta_1 \times \{0,1\}).$$

Geometrically, this simplicial subset of the square looks like

$$\begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ & & \uparrow \\ (0,0) & \longrightarrow & (1,0). \end{array}$$

To prove $A \hookrightarrow \Delta_1^2$ is anodyne, we have two options: either directly explain how to solve lifting problems against it, or try to build it out of compositions, retracts, and pushouts from horns. The second approach seems more likely to generalize well, so let's try that.

Can we build $\Delta_1 \times \Delta_1$ out of $A$ just by attaching simplices along horns? Fortunately, yes!

Define $A'$ as the pushout

$$\begin{array}{ccc} \Lambda_2^1 & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Delta_2 & \longrightarrow & A', \end{array}$$

where the map $\Lambda_2^1 \to A$ comes from the $(0,0) \to (1,0)$ and $(1,0) \to (1,1)$ edges.

Then $A'$ looks like

$$\begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ & \nearrow & \uparrow \\ (0,0) & \longrightarrow & (1,0), \end{array}$$

where the bottom triangle is a filled 2-simplex.

But now we just attach one more 2-simplex, along the $\Lambda_2^2$-horn with vertices $(0,1), (1,1), (0,0)$, to get the simplicial square. Thus $i_1 \boxtimes j$ is strongly anodyne: it is the composition

$$A \hookrightarrow A' \hookrightarrow \Delta_1^2,$$

where $A \to A'$ and $A' \to \Delta_1^2$ are both strongly anodyne as they are pushouts of horn inclusions.

With a little more combinatorial work, we can make this idea work in higher dimensions.

### A.5.2 Finishing the simplest horn

**Lemma A.5.3.** *Let $i : A \hookrightarrow A'$ be an inclusion of simplicial sets. Then, for $j : \{1\} \hookrightarrow \Delta_1$ the horn inclusion, the Leibniz product $i \boxtimes j$ is strongly anodyne.*

*Proof.* By Ideation A.5.1, it is enough to prove every $i_n \boxtimes j$ is strongly anodyne. Define

$$A_n := (\Delta_1 \times \partial\Delta_n) \coprod_{\{1\}\times\partial\Delta_n} (\{1\} \times \Delta_n).$$

Set $P_n = \Delta_1 \times \Delta_n$ (the $P$ is for prism).

Paralleling the strategy of Ideation A.5.2, we explain how to build the prism $P_n$ from $A_n$.

We can view $\Delta_1 \times \Delta_n$ as the nerve of the poset (or category) $[1] \times [n]$, and in particular each nondegenerate simplex is determined by its vertices. For each $0 \leq i \leq n$ let $\sigma_i$ be the $(n+1)$-simplex of our prism spanned by $(0,0), (0,1), ..., (0,i), (1,i), (1,i+1), ..., (1,n)$.

These $(n+1)$-simplices $\sigma_0, \sigma_1, ..., \sigma_n$ are all the nondegenerate $(n+1)$-simplices of $P_n$. We are going to attach them to $A_n$ one at a time to build $P_n$.

Put $B_{-1} = A_n$. We inductively define simplicial subsets $B_i$ of $P_n$ by $B_i = B_{i-1} \cup \sigma_i$. Observe then that the inclusion $B_{i-1} \hookrightarrow B_i$ fits into a pushout diagram

$$\begin{array}{ccc} \Lambda^{i+1}_{n+1} & \longrightarrow & B_{i-1} \\ \downarrow & & \downarrow \\ \Delta_{n+1} & \xrightarrow{\sigma_i} & B_i. \end{array}$$

To see this, it suffices to check that the intersection of $B_{i-1}$ and $\sigma_i$ (taken in $P_n$) is the horn $\Lambda^{i+1}_{n+1}$. In other words, we must show that every non-degenerate $n$-simplex of $\sigma_i$ lies in $B_{i-1}$ except for the non-degenerate $n$-simplex opposite to the $(i+1)^{\text{st}}$ vertex of $\sigma_i$.

For this, we start by understanding the non-degenerate $n$-simplices of $P_n$. These are determined by their vertices; most are contained in $A_n$, with the exception of the simplices

$$\tau_i = \{(0,0), ..., (0,i), (1,i+1), ..., (1,n)\},$$

for $0 \leq i \leq n$. These are the simplices whose projection to $\Delta_n$ is non-degenerate, but whose projection to $\Delta_1$ is not the constant simplex 1.

For $i > 0$, the face of $\sigma_i$ opposite $(0,j)$, for $j < i$, is contained in $\Delta_1 \times \partial\Delta_n$, and hence in $A_n$. Similarly, for $j > i$, the face opposite $(1,j)$ lies in $A_n$. The face opposite $(0,i)$ is $\tau_{i-1}$, and the face opposite $(1,i)$ is $\tau_i$. The $i = 0$ case is left to the reader.

Notice that $\sigma_i$ thus has two faces with the potential to not lie in $B_{i-1}$: $\tau_{i-1}$ and $\tau_i$. But $\tau_{i-1}$ is a face of $\sigma_{i-1}$, and hence lies in $B_{i-1}$; thus the only new non-degenerate $n$-simplex in $B_i$ is $\tau_i$, explaining our pushout from before.

As each $B_{i-1} \hookrightarrow B_i$ is therefore strongly anodyne, we deduce that

$$A_n \hookrightarrow P_n$$

is a composition of strongly anodyne maps and hence strongly anodyne. □

### A.5.3 Harder horns

To end, we just need to go beyond the case of prisms. For this, we copy the strategy of Kerodon's tag 014D, which gives a clever retraction.

**Lemma A.5.4.** *For each $0 < k \leq n$, the horn inclusion $j : \Lambda^k_n \hookrightarrow \Delta_n$ is a retract of the inclusion*

$$(\Delta_1 \times \Lambda^k_n) \coprod_{\{1\}\times\Lambda^k_n} (\{1\} \times \Delta_n) \hookrightarrow \Delta_1 \times \Delta_n.$$

*Proof.* Set

$$A = (\Delta_1 \times \Lambda^k_n) \coprod_{\{1\}\times\Lambda^k_n} (\{1\} \times \Delta_n).$$

We need to find a retraction of $\{0\} \times \Delta_n \hookrightarrow \Delta_1 \times \Delta_n$ which sends $A$ to $\Lambda^k_n$. Fortunately, this is easy.

Maps

$$r : \Delta_1 \times \Delta_n \to \Delta_n$$

are the same as monotone functions $[1] \times [n] \to [n]$ (as the nerve functor from posets to sSet is fully faithful, as a consequence of $N : \mathrm{Cat} \to \mathrm{sSet}$ being fully faithful). We desire a monotone function which sends $\{0\} \times \Delta_n$ to $\Delta_n$, and which sends $A$ to $\Lambda^k_n$. The first constraint forces us to put

$$r(0,0) = 0, r(0,1) = 1, ..., r(0,n) = n.$$

It's a little overkill, but we can get $\{1\} \times \Delta_n$ to map into $\Lambda^k_n$ by contracting it a bit to

$$r(1,0) = k, ..., r(1,k-1) = k, r(1,k) = k, r(1,k+1) = k+1, ..., r(1,n) = n.$$

The image of $\{1\} \times \Delta_n$ under this $r$ is the $n$-simplex of $\Delta_n$ spanned by the vertices $k, k, ..., k, k+1, k+2, ..., n$. Thus, as long as $k > 0$, this is a degenerate simplex lying in $\Lambda^k_n$.

Similarly, we can check $r(\Delta_1 \times \Lambda^k_n) \subseteq \Lambda^k_n$, and so we get a commutative diagram

$$\begin{array}{ccccc} \{0\} \times \Lambda^k_n & \longrightarrow & A & \xrightarrow{r} & \Lambda^k_n \\ \downarrow & & \downarrow & & \\ \{0\} \times \Delta_n & \longrightarrow & \Delta_1 \times \Delta_n & \xrightarrow{r} & \Delta_n, \end{array}$$

which exhibits the horn inclusion as a retract. □

**Lemma A.5.5.** *Let $i : A \hookrightarrow A'$ be an inclusion, and $j : \Lambda^k_n \hookrightarrow \Delta_n$ a horn inclusion with $k > 0$. Then $i \boxtimes j$ is strongly anodyne.*

*Proof.* As the class $T$ of morphisms $j$ such that $i \boxtimes j$ is strongly anodyne is stable under retracts, by Lemma A.5.4 it suffices to prove that

$$j \boxtimes j' : (\Delta_1 \times \Lambda^k_n) \coprod_{\{1\}\times\Lambda^k_n} (\{1\} \times \Delta_n) \hookrightarrow \Delta_1 \times \Delta_n$$

belongs to $T$, where $j' : \{1\} \hookrightarrow \Delta_1$ is the standard inclusion.

Recall that the Leibniz product is associative, and so

$$i \boxtimes (j \boxtimes j') = (i \boxtimes j) \boxtimes j'.$$

By Lemma A.5.3, this Leibniz product is strongly anodyne, as desired. □

## A.6 Concluding

Finally, we can finish our proof of Theorem 4.8.8.

**Theorem A.6.1** (Restatement of Theorem 4.8.8)**.** *Let $i : A \hookrightarrow A'$ be an inclusion, and $j : B \hookrightarrow B'$ be a strongly anodyne inclusion. Then $i \boxtimes j$ is strongly anodyne.*

*Proof.* Let $T$ be the class of all inclusions $j$ for which $i \boxtimes j$ is strongly anodyne. By Lemma A.4.1 and the definition of strongly anodyne, it suffices to prove that $T$ contains all horn inclusions.

By Lemma A.5.5, this class $T$ contains all horn inclusions $\Lambda^k_n \hookrightarrow \Delta_n$ with $k > 0$.

To get the $\Lambda^0_n \hookrightarrow \Delta_n$ horn, just note that it is the opposite to $\Lambda^n_n \hookrightarrow \Delta_n$, and as

$$(i \boxtimes j)^{\mathrm{op}} = i^{\mathrm{op}} \boxtimes j^{\mathrm{op}},$$

we conclude as long as we show that opposites of strongly anodyne morphisms are still strongly anodyne.

And indeed, let $T'$ be the class of all morphisms whose opposite is strongly anodyne. By Proposition A.2.4, this family contains all horn inclusions; it also clearly contains all isomorphisms. As taking opposites is an involution of sSet, this class $T'$ is stable under composition, $\omega$-composition, pushouts, retracts, arbitrary coproducts. Thus $T'$ contains all strongly anodyne morphisms, and so we conclude. □